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AN ELEMENTARY COURSE

OF

PRACTICAL MATHEMATICS,

FOR THE USE OF SCHOOLS.

BY JAMES ELLIOT,

AUTHOR OF "A COMPLETE TREATISE ON PRACTICAL GEOMETRY AND MENSURATION,"
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AND ON PLANE TRIGONOMETRY," ETC.

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PREFACE.

THE Courses of Mathematics generally used, while insufficient to give the student a thorough knowledge of each department, are yet too difficult for beginners. There is, perhaps, no compilation of that character, which can be said to be in every part rudimentary. We have, for instance, Arithmetic of Sines, the modes of computing Logarithms,* and other comparatively abstruse subjects, included in treatises designed for beginners,—forming a serious impediment to their progress, and to the attainment of knowledge much more useful to them. At the same time, in none of such works will the student obtain information sufficient to carry him to the full extent of any one subject. To attempt, indeed, to give such an amount of matter in the compass of a single volume, is to aim at an impossibility.

There are two distinct objects for which an elementary course of Mathematics may be required in schools:—*one*, to supply all that is wanted by the great majority of scholars, by those who study the subject merely as one branch of a general education; and *the other*, to serve as an *intro-*

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* Not the modes of computing *by* Logarithms—which belong, *properly*, to an elementary work—but the rules for the calculation of the tables themselves, with which a beginner has nothing whatever to do.

ductory work to those who are likely, from taste or professional requirements, to devote themselves peculiarly to the study. These objects are, happily, quite compatible with each other, and both may be answered by the same kind of compilation. It is equally desirable for both, that the manual employed should contain nothing that is intricate. For the one class of learners, such matter is not required at all; and for the other, it ought to be reserved to a subsequent stage of their progress. When introduced into an elementary work, it is not only useless, but a positive obstruction: the teacher has great difficulty in separating it from the more useful matter; and, when it is not so separated, the pupil finds himself embarrassed with subjects beyond his comprehension, and becomes dissatisfied with the study altogether. It is possible, no doubt, by a distinction of type, and a very careful arrangement, to make the same volumes serve both for beginners and for the more advanced. This object has been effected in the Author's Treatises on "Practical Geometry and Mensuration," and on "Logarithms and Plane Trigonometry." But a complete course on such a system would be too voluminous and expensive for the greater number of learners, and contain much that they would never require.

Another alternative is an easy course for general use, and also for the first practice of those who may intend ultimately to advance further. In the case of the latter, an objection might be made to commencing with one text-book, and finishing with another, unless both proceeded on the same system,—unless, in fact, the work subsequently used could be made to harmonise exactly with that which served for an introduction. *That*, then, is precisely the desideratum which is proposed to be supplied by means of this volume,* in conjunction with those alluded to above,—as far, at least, as the extent of the subjects which the latter comprehend. Under the heads mentioned, the material of this volume will be, as nearly as possible, identical with the first course of the more extended treatises, so that the student may pass easily from the one to the other.* In Algebra, indeed, the Author's plan is not complete, and is not intended to be completed. In that department it is not necessary. We already possess many excellent books on the subject for advanced students; and the methods employed in the first stages are so much alike, that it is easy to pass from this to any one of the more comprehensive works.

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* Including the Parts subsequently to appear.

In order to limit the size and price of this volume to those of an ordinary school-book, without injudicious condensation and smallness of type, it is needless to say that it has been found necessary merely to give the practice, leaving theory to the explanations of the teacher, where he may think elucidation useful, and to the intelligence of the pupil. Independently of the reason just assigned, this is perhaps no evil in a work intended for beginners. In Geometry, practice should, *to a certain extent*, precede theory. To place Euclid before a pupil who has never handled a pair of compasses, is as likely to stultify as to enlighten him. Algebra, in its ordinary processes, usually carries its own demonstrations with it, or merely requires a hint for their development: its rules are often self-supported; and, in other cases, nearly so. In all departments, a *vivâ-voce* explanation is more readily understood by the young than a written one, and can be modified by the teacher to suit the various degrees of capacity. In fact, theoretical explanations in elementary works are seldom read except by the teacher. Far be it from the writer of this to decry theory. Without it, no great advancement can be made in any science—and least of all, perhaps, in Mathematics; but it is not necessary that the attainment of any science should be *begun* by theory; and it is still less necessary that, *for the young*, theoretical explanation should always be perfect demonstration, and in print. Much may be done towards demonstration by mere arrangement. Half the demonstration is accomplished, when the rules are arranged in that order in which they naturally flow from each other.

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* When the pupil is decidedly intended to acquire more than an elementary knowledge of Mathematics, it is no doubt better at once to commence with the Author's larger volumes, on the heads mentioned; and it is probably preferable to do so in every case in which the slight difference of expense, or the size of the volumes, is not objected to. The publication of this in separate parts will leave it perfectly open to pursue either plan, and *that* either wholly or partially.

The Author would recommend to the teacher the following plan for every student of Mathematics, whether his object be to acquire a merely elementary knowledge of the subject, or to pursue it to a greater extent. Let him *commence* with *an easy practical course*, the teacher pointing out the Theory as he sees proper,—sometimes by mere hints, sometimes by familiar illustrations, sometimes by rigid demonstration. It is not by any means necessary, that the scholar should finish the practical course before commencing the theoretical. On the contrary, it will be much better, when he has advanced a short way with the former (when he has finished Practical Geometry, for instance, in this course), to commence Euclid's Elements, or any similar work, and to go on with the two together. If he uses this volume, and is inclined, when he has finished it, to extend his knowledge of Practical Mathematics, let him then take up the Author's more diffuse Treatises, in which he will find all that he has acquired, carefully distinguished from that which still lies before him, and the latter harmonising in system and arrangement with the former.

There will probably be a difference of opinion among teachers regarding the *order* in which the different subjects in this volume should be studied. Some will prefer commencing with Practical Geometry and Mensuration, leaving Algebra to a later stage, as a more difficult and less generally useful subject. There is good ground for that preference; and it would have been adopted in the arrangement of this volume, had it not been that a knowledge of the signs and modes of expression used in Algebra is necessary to understanding the formulæ used in Mensuration. Nothing more of Algebra, however, is required, for that purpose, than the definitions, and some parts, perhaps, of the five chapters which immediately follow them. The teacher may, therefore, if he think proper, commence with the first six chapters of Algebra,—then Practical Geometry and Mensuration,—next the remainder of the Algebra, followed by Logarithms, Plane Trigonometry, and the remaining subjects; or, if he prefer, he may complete the department of Algebra, before commencing the others.

EDINBURGH, 1*st October* 1850.

PRACTICAL MATHEMATICS.

PART FIRST.

ALGEBRA.

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ALGEBRA.

CHAPTER I.

DEFINITIONS AND EXPLANATIONS.

1. WHILE Arithmetic represents numbers by figures, Algebra represents them by letters.

2. While the figures of Arithmetic represent only *particular* numbers (each figure, or combination of figures, denoting one number only), the letters of Algebra may represent *any* numbers, or classes of numbers. Algebra is, therefore, sometimes called *Universal Arithmetic.*

3. The letters of Algebra represent not merely numbers but also quantities. The quantities, however, when subjected to Algebraical processes, are regarded as numbers, some other quantity being taken as the unit; and even when the letters represent numbers, they are often called by the general name of *Quantities.* The words *Number* and *Quantity*, in the language of Algebra, are therefore synonymous.

4. While the figures of Arithmetic represent only *known* numbers, those of Algebra sometimes represent numbers *not known.* The letters at the commencement of the Alphabet are used to denote known numbers, as a, b, c; those at its close, to denote numbers unknown, as x, y, z.

5. Both Arithmetic and Algebra employ signs to connect the numbers, and to express the operations to which they are subjected, but Algebra employs such signs much more extensively.

6. $+$ (named *plus*) is the sign of addition. Thus, $3+2$ means 3 added to 2, or 5.

7. $-$ (*minus*) is the sign of subtraction, and means that the quantity following the sign is to be taken from that which precedes it. Thus, $5-2$ means 2 taken from 5, or 3.

8. When the sign + is prefixed to any number, and no other number stands before it to which it can be added (as + 7), the number to which the sign is prefixed is called a *Positive Number*. When the sign − is similarly placed, as (− 7), the number to which it is prefixed is (along with the sign), called a *Negative Number;* that is,—a number less than nothing. Even when such numbers or quantities are connected with other numbers by their signs, they may still be considered by themselves without reference to such other numbers, and then also described as positive or negative numbers.

All actual numbers, such as we usually speak of, and subject to the ordinary operations of Arithmetic, are positive numbers; but we cannot form an idea of negative numbers, unless relatively to positive numbers, of which they are the opposite. Thus, if + 5 mean 5 Pounds of property, then − 5 will mean 5 Pounds of debt; if + 3 is used for 3 feet of height, then − 3 stands for 3 feet of depth. When a number stands without either of these two signs, it is understood to be positive.

9. The signs + and − themselves are named *Positive* and *Negative Signs*.

10. ± (*plus or minus*) indicates that the second quantity is either to be added to, or subtracted from, the first.

Exercise. What is the amount of each of the following expressions? $7+5$; $3+4+6$; $6+4+3$; $7-5$; $8+9-15$; $8-15+9$; $5-7$; 10 ± 2.

11. × (*into*) is the sign of multiplication. Thus, 3×2 means 3 times 2, or twice 3, or 6. Instead of this sign we often employ a point, as 3.2; or, when letters are used, or figures and letters conjointly, we simply write them together, as ab, or $7bc$.

12. ÷ (*by*) is the sign of division, the dividend preceding it, and the divisor following. Thus, $21 \div 3$ means 21 divided by 3,—that is, 7. The same thing is more frequently expressed by writing the divisor below the dividend in the form of a fraction, as $\frac{21}{3}$.

Exercise. What is the simple value of each of the following expressions? 7×9; 9×7; $5\times 8\times 2$; 8.5.2; $8\div 2$; $\frac{12}{4}$; $\frac{9\times 5}{3}$; and $9.5\div 3$.

Exercise. If the letter a denote 10; b, 6; c, 2; d, 1; *and* e, *0, what* are the values of the following expressions? $b+c$; $b-c$; $b \pm c$; $a-e$; $a+b+c+d+e$; $a-b+c-d$; $a \times b$; $b.c$; abc; $5ac$; $abcde$; $a \div c$; $\frac{b}{c}$; $\frac{d}{c}$; $\frac{a+b}{c}$; $\frac{a-b}{c}$; and $\frac{ac-2d}{b}$.

13. $=$ (*equal to*) is the sign of equality. Thus, $5+3=8$, $10 \div 2 = 5$.

14. $: \, :: \, :$ is the sign of proportion. Thus, $a : b :: c : d$, is read, As a is to b, so is c to d.*

15. The sign $\therefore$ signifies *therefore*, and $\because$ signifies *because*.

16. When a number is the product of two or more numbers multiplied together, the latter numbers are called the *Factors* of the former. Thus, 7 and 3 are the factors of the number 21; a, b, and c are the factors of the quantity abc.

Exercise. What are the factors of the number 30?

17. A *prime Number*, or *prime Factor*, is that which cannot be resolved into other factors; as 2, 7, a, or $x+y$.

18. When the first of two or more factors is a figure, and the others letters, as $7ab$, the figure is called the *Co-efficient* of the letters. When no co-efficient is expressed, it is understood to be 1. Sometimes, also, a letter is taken as a co-efficient, especially when that letter represents a known number, and the letter, to which it is attached, an unknown number. Thus, in the expression mx, m may be regarded as the co-efficient of x.

19. When all the factors are alike, as ccc, the product is called a *Power* of the single factor, and the latter is called a *Root* of the former. Thus, $bbbb$ is a power of b, and d is a root of ddd. Thus, also, 27 is a power of 3, and 3 is a root of 27; for $27 = 3.3.3$.

20. The *number* of equal factors is called the *Exponent* or *Index* of the power or root, and by that number the power or root is described. Thus, 16 is called the fourth power of 2; and 2 the fourth root of 16, the index being 4; for $16 = 2.2.2.2$. Thus, also, $bbbbb$ is the fifth power of b, and the latter is the fifth root of the former, the index being 5.

21. The *Square* of any number is its second power, and the *Cube* is its third power. In like manner the *Square Root* and the *Cube Root* are the second and third roots.

22. Any power of a number is expressed briefly by

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* Many of these definitions belong to Arithmetic as well as to Algebra; but it is necessary to give a synopsis of the whole.

writing the exponent over the number, as, 4^2, a^6,—meaning the second power of 4, and the sixth power of a. When no exponent is expressed, it is understood to be 1, any number being its own first power.

23. A root is expressed by prefixing the *Radical* Sign, $\sqrt{\ }$, with the exponent over it, as, $\sqrt[5]{n}$,—meaning the fifth root of n. When the radical sign occurs without an exponent expressed, the square root is meant, as, $\sqrt{9} = 3$.

EXERCISE. What are the values of 12^2, 5^3, 2^5, $\sqrt{64}$, $\sqrt[3]{27}$, $\sqrt[4]{16}$?

24. Any single quantity, as, a, or the combination of any number of quantities, by multiplication or division, as, $b \times c$, or, $2abc$, or, $\frac{ab}{cd}$, is called a *Single Term.*

25. *Like Terms* are those made up of the same letters similarly combined, although their co-efficients may differ. Thus, $3a$ and $7a$ are like terms; $\frac{b}{c}$, $2\frac{b}{c}$, and $7\frac{b}{c}$, are also like terms.

26. *Unlike Terms* are those made up of different letters, or of the same letters so combined as to give different values. Thus, a, b, ab, $\frac{a}{b}$, and $\frac{b}{a}$, are all unlike terms; but ab and ba are not unlike, for their values are necessarily the same.

27. A *Simple Quantity* is that which consists of a single term, as $\sqrt{3xy}$.

28. A *Compound Quantity* consists of two or more terms connected together by the signs + or −; as, $a + b - c$.

29. If a compound quantity consist of only two terms, it is called a *Binomial;* as, $x - y$.

30. Terms following each other, and changing according to any given law, are called a *Series;* as, 2, 3, 4. When the series has no termination it is called an *Infinite Series;* as, 2, 4, 8, 16, 32, &c., *ad infinitum.*

31. A parenthesis or line, connecting the terms of a compound quantity, is called a *Vinculum;* as, $\overline{m + n}$, or, $(a + b - c)$, or, $\left\{\frac{ax}{3} - \frac{by}{2}\right\}$. It indicates that the compound quantity so enclosed is to be taken as a whole, and no regard paid to the separate terms of which it consists.

32. The vinculum is sometimes required to connect not only the different parts of a compound quantity, but also the component parts of the same term, when these parts

are separated by the sign of multiplication or of division, and yet all affected by one exponent of a power, or one radical sign; as, $(21 \div 7)^3$, or, $\sqrt{\left(\frac{a}{c} \times \frac{m}{n}\right)}$. When a term is made up of several simple factors written together, without the sign of multiplication interposed, a vinculum is not required to bring the whole term under the influence of the same radical sign; but it is required to bring them under the influence of one exponent of a power. Thus we say, $\sqrt{xy}$, $\sqrt[3]{2ab}$, and $(2ab)^5$, when the radical sign and exponent affect the whole term, xy or $2ab$. But if these signs are applied only to the factor next them, we write the terms thus, $\sqrt{x}.y$, $\sqrt[3]{2} \times ab$, and $2ab^5$, or $2a \times b^5$, or $2a.b^5$.

33. The line between the terms of a fraction serves as a vinculum both to the numerator and to the denominator, when compound quantities. The separating line has the same effect when division is expressed in the fractional form; as, $\frac{a+b}{a-b}$.

NOTE. Additional definitions will be given under the heads to which they *particularly* belong.

CHAPTER II.

ADDITION.

Addition, in Algebra, is the uniting of various quantities into one simple quantity, or into a compound quantity in its simplest form.

In Arithmetic the addition of a number always increases that to which it is added. Not so in Algebra; for the addition of negative quantities neutralizes an equivalent of positive quantities, so that the addition of a negative quantity is the same as the subtraction of a positive quantity.

PROBLEM I.

To add simple Quantities together.

CASE 1. *When the quantities are like* and with like signs.

RULE. Add the co-efficients together by common Arithmetic, and set down the sum, followed by the common letter or letters, and preceded by the common sign, + or −.

Thus, $4ab$ added to $3ab$ makes $7ab$, and $-6x$ added to $-4x$ makes $-10x$.

NOTE. When there are many quantities to be added, they should be arranged in columns as in Arithmetic.

EXERCISE 1. Add together $3a$, $15a$, $7a$, and $9a$.

Ans. $34a$.

2. Add into one sum $+6b^2$, $+3b^2$, $+b^2$, and $+4b^2$.

Ans. $+14b^2$.

3. What is the sum of $-6c$, $-12c$, and $-11c$?

Ans. $-29c$.

4. Add together $-23a\frac{x}{y}$, $-67a\frac{x}{y}$, $-34a\frac{x}{y}$, and $-108a\frac{x}{y}$.

Ans. $-232a\frac{x}{y}$.

CASE 2. *When there are only two like quantities with unlike signs.*

RULE. Subtract the smaller co-efficient from the greater, and set down the remainder, preceded by the sign of the greater, and followed by the common letter or letters.

Thus, $+7x$ added to $-3x$ makes $+4x$, and $-7x$ added to $+3x$ makes $-4x$.

EXERCISE 1. Add together $+10m$ and $-6m$.

Ans. $+4m$.

2. Add $-49n$ to $+25n$. Ans. $-24n$.

3. What is the sum of $+253\sqrt{xy}$ and $-741\sqrt{xy}$?

Ans. $-488\sqrt{xy}$.

CASE 3. *When there are several like quantities with unlike signs.*

RULE. Add all the positive quantities into one sum, and all the negative into another, by Case 1. Then add these two sums together, by Case 2.

NOTE. When the quantities are numerous, we may, if we choose, place the positive quantities in one column and the negative in another.

EXAMPLE. Add together $+7c$, $-5c$, $+c$, $+15c$, $-4c$, and $-10c$.

$$\begin{array}{rr} +\ 7c & -\ 5c \\ +\ 1c & -\ 4c \\ +15c & -10c \\ \hline +23c & -19c = 4c, \text{ Ans.} \end{array}$$

EXERCISE 1. Add together $5a$, $-3a$, $-10a$, and $18a$.

Ans. $+10a$.

2. Collect into one sum, $-5bx$, $+7bx$, $-19bx$, $+3bx$, and $-12bx$. Ans. $-26bx$.

3. What is the sum of $+64m^2$, $-17m^2$, $+95m^2$, $+23m^2$, $-56m^2$, $-49m^2$, $-31m^2$, $+27m^2$, $-35m^2$? Ans. $+21m^2$.

4. Add together $19\sqrt{xy}$, $-45\sqrt{xy}$, $+38\sqrt{xy}$, $-39\sqrt{xy}$, $+27\sqrt{xy}$. Ans. 0.

Case 4. *When all the quantities are unlike.*

Rule. All that can be done is to place them together, connected by their proper signs.

Thus, if we are required to add into one sum, $3a$, $-2b$, and $\sqrt{c}$, we cannot put that sum into a simpler form than $3a-2b+\sqrt{c}$.

Exercise. Add together a, $-b$, $+\sqrt{ab}$, and $-\frac{b}{a}$.

Case 5. *When there are several quantities, partly like and partly unlike.*

Rule. Add together the like quantities, as in Cases 1, 2, and 3, and connect the sums with each other (if more than one), and with the unlike quantities, as in Case 4.

Example. What is the sum of $+7a$, $-13b$, $+5x$, $+2a$, $-7y$, $-2\sqrt{xy}$, and $+6b$?

$$\begin{array}{rl} & +7a-13b \\ & +2a+\ \ 6b+5x-7y-2\sqrt{xy} \\ \hline \text{Ans.} & +9a-\ \ 7b+5x-7y-2\sqrt{xy}. \end{array}$$

Exercise 1. Add together $7a$, $4b$, $5c$, $6a$, $3b$, and $10a$. Ans. $23a+7b+5c$.

2. Express in the simplest form $3a^2+9-7a+5\sqrt{b}-16+11a-13a^2-12\sqrt{ab}-23$. Ans. $4a-10a^2+5\sqrt{b}-12\sqrt{ab}-30$.

3. Simplify the expression $17a^2-34b^2-51c^2+13a^2-26ab+13b^2-6a^2+12ac-6c^2+55b^2-110bc+55c^2$. Ans. $24a^2-26ab+34b^2+12ac-110bc-2c^2$.

Problem II.

To add compound Quantities together.

Rule. Add the simple quantities of which they consist, separately.

Example 1. Add together $3a-7b$, $5a+4b$, and $-2a-2b$.

$$\begin{array}{rl} & +3a-7b \\ & +5a+4b \\ & -2a-2b \\ \hline \text{Ans.} & 6a-5b. \end{array}$$

EXERCISE 1. Add together $7a^2-3b^2$, $5a^2-12b^2$, and $4a^2-4b^2$. Ans. $16a^2-19b^2$.

2. Add together $3a-4\sqrt{bc}$, $-5a+8\sqrt{bc}$, $-10a-10\sqrt{bc}$, and $7a-6\sqrt{bc}$. Ans. $-5a-22\sqrt{bc}$.

3. Add into one sum, $4x^2+4xy+4y^2$, $7x^2-7xy-3y^2$, $-8x^2-8xy+5y^2$, $-3x^2+7xy-8y^2$, and $x^2+4xy+y^2$.

Ans. x^2-y^2.

4. What is the sum of $x^3+3x^2y+3xy^2+y^3$, $5x^3-15x^2y+15xy^2-5y^3$, $10x^3+10y^3$, $17x^2y-17xy^2$, and $-13x^3+39x^2y-39xy^2+13y^3$?

Ans. $3x^3+44x^2y-38xy^2+19y^3$.

NOTE 1. It will sometimes be necessary to re-arrange the terms, in order that each may stand in its proper column.

EXAMPLE 2. Add together $3a+6x-7y$, $2a-7x+5b$, and $x+b+y$.

$$\begin{array}{rrrr}
3a & & +6x & -7y \\
2a & +5b & -7x & \\
 & 1b & +1x & +1y \\
\hline
\text{Ans. } 5a & +6b & * & -6y.
\end{array}$$

EXERCISE 5. Find the sum of $a^2+b^2+c^2$, $-6ab-6ac-6bc$, $25a^2-10ab+b^2$, $a^2-8ac+16c^2$, and $-4b^2+6bc-9c^2$. Ans. $27a^2-16ab-2b^2-14ac-8c^2$.

6. Required, the amount of $7x^2-16+5y$, $24-\sqrt{xy}-15x^2+3xy$, and $3x^2-14+28y$.

Ans. $-5x^2+3xy-\sqrt{xy}+33y-6$.

7. Collect the following quantities into one sum:— $15a^2b+15ab^2-16ac^2+15b^2c$, $a^3-3ab^2-3ac^2-b^3+12c^3$, $-5a^3+20b^2c+15bc^2-6a^2c$, and $-12ab^2-12bc^2+16ac^2+24a^2b-8a^2c-15b^2c$.

Ans. $-4a^3+39a^2b-b^3-14a^2c-3ac^2+3bc^2+12c^3$.

NOTE 2. When compound quantities are enclosed in vincula, they are often added together as if they were simple quantities, especially if several of them are alike. Powers and roots of compound quantities are also regarded as simple quantities, and added as such. So are fractions whose terms are compound.

EXAMPLE 3. What is the sum of $+3(a-b)$, $-5(a-b)$, and $+7(a-b)$?

$$\begin{array}{rr}
 & +3(a-b) \\
 & -5(a-b) \\
 & +7(a-b) \\
\hline
\text{Ans.} & +5(a-b).
\end{array}$$

EXERCISE 8.—Find the sum of $+7(a+x)$, $-13(a+x)$,

$-22(a+x)$, $+55(a+x)$, $-36(a+x)$, $+18(a+x)$, and $-21(a+x)$. Ans. $-12(a+x)$.

9. Add together $3(a+b)$, $-5(a-b)$, $4(a-b)$, $-6(a+b)$, $7(a+b)+7(a-b)$, and $(a-b)-(a+b)$.
Ans. $3(a+b)+7(a-b)$.

10. Add together $5(x-y)^2$, $-7(x-y)^2$, $+24(x-y)^2$, and $-13(x-y)^2$. Ans. $9(x-y)^2$.

11. Collect into one sum $4\sqrt{(x^2+y^2)}$, $5\sqrt{(x^2-y^2)}$, $-7\sqrt{(x^2-y^2)}$, $-6\sqrt{(x^2+y^2)}$, $+20\sqrt{(x^2+y^2)}$, and $-8\sqrt{(x^2-y^2)}$. Ans. $18\sqrt{(x^2+y^2)}-10\sqrt{(x^2-y^2)}$.

12. Add $369\,\dfrac{a^2+b^2}{a^2-b^2}$ to $-484\,\dfrac{a^2+b^2}{a^2-b^2}$.

Ans. $-115\,\dfrac{a^2+b^2}{a^2-b^2}$.

CHAPTER III.

SUBTRACTION.

General Rule. Change the sign, or signs, of the subtrahend, and add it (thus changed) to the minuend.

Note. Instead of actually changing the signs of the subtrahend, it is better to write it anew with the signs altered. After some practice, however, it will be found sufficient not to change the signs at all, but merely to suppose them changed; and even that, after a little practice, will seldom be necessary.

Example. From $7a-16b+4c-5d+e-f$ subtract $a-9b+5c-8d+12e-13f$.

$$\begin{array}{r}
+7a-16b+4c-5d+\ 1e-\ 1f \\
-1a+\ \ 9b-5c+8d-12e+13f \\
\hline
\text{Ans. } 6a-\ \ 7b-\ c+3d-11e+12f.
\end{array}$$

Exercise 1. From $36x$ take $15x$. Ans. $+21x$.

2. From $15x$ subtract $36x$. Ans. $-21x$.

3. From $-3y$ subtract $-17y$. Ans. $+14y$.

4. From $-17y$ subtract $-3y$. Ans. $-14y$.

5. From $m+12n$ take $15m+7n$. Ans. $-14m+5n$.

6. From $17m+3n$ take $-11m-12n$. Ans. $28m+15n$.

7. From $-6a^2-24b^2$ take $18a^2+9b^2$.
Ans. $-24a^2-33b^2$.

8. From $-15\sqrt{ab}-7\sqrt{ac}$ subtract $-3\sqrt{ab}-12\sqrt{ac}$.
Ans. $-12\sqrt{ab}+5\sqrt{ac}$.

9. Subtract $a^3-3a^2b+3ab^2-b^3$ from $a^3+6a^2b+12ab^2+8b^3$. Ans. $9a^2b+9ab^2+9b^3$.

10. Subtract $3a^5-9a^4+15a^3-24a^2+30a-45$ from $a^5+4a^4+20a^3+25a^2+20a-45$.
Ans. $-2a^5+13a^4+5a^3+49a^2-10a$.

11. Subtract $19a^2-24b^2+27ax+13y^2$ from $33a^2+7ac+46ax-12y^2-7z^2$.
Ans. $14a^2+24b^2+7ac+19ax-25y^2-7z^2$.

12. Subtract $4x^2y-7xy^2+5xz^2-24yz^2-xyz$ from $4x^2y-7x^2z+12y^2z-24yz^2+xyz$.
Ans. $7xy^2-5xz^2-7x^2z+2xyz+12y^2z$.

13. Take $12(a+x)^2-63(b-y)^2$ from $63(a+x)^2-12(b-y)^2$. Ans. $51(a+x)^2-51(b-y)^2$.

14. Take $19(a-b)-19(a+b)$ from $(a+b)+(a-b)$.
Ans. $20(a+b)-18(a-b)$.

CHAPTER IV.

THE VINCULUM.

The vinculum, explained in Definitions 31 and 32, is of frequent and important employment in Algebra. When it connects the terms of a compound quantity, its use requires particular explanation, since no sign is so liable to be the source of mistakes.

When a vinculum is already in an expression, and to remain in it, its use is very simple: we then merely treat the quantity within the vinculum as a whole, and pay no regard to its component parts, precisely as if the vinculum and the quantity within it were one letter. We treat such quantities as sealed parcels, of which we may count the numbers, and subject those numbers to any operations of Arithmetic, as parcels, without knowing what they contain. The removing of the vincula is like the uncovering of the parcels,—their contents are separated, and then their re-arrangement requires peculiar rules. The enclosing of separate quantities within vincula also requires rules. The following will suffice :—

Rule 1. Any sign, immediately preceding a vinculum, is applied to the whole quantity within the vinculum, and not to any one of its terms or factors. Thus, $-(a+b)$ is entirely different from $(-a+b)$. Thus, also, $\surd(9+7)=4$, but $\surd 9+7=10$.

Rule 2. When no sign, within the vinculum, precedes the first term, that term is understood to have the sign +. Thus $-(a+b)$ is the same as $-(+a+b)$.

Exercise 1. What are the values of $+(8+6)$, $+(8-6)$, $-(8+6)$, and $-(8-6)$?

2. What are the values of $8+2^2$, $(8+2)^2$, $8-2^3$, $(8-2)^3$, $\sqrt{4}+5$, and $\sqrt{(4+5)}$?

Rule 3. Since the sign +, prefixed to a vinculum containing a compound quantity, implies that the whole compound quantity is to be added, *the removal of a vinculum*, when preceded by +, will not require a change of any of the signs within the vinculum. Thus, $8+(6-2)$ is exactly equivalent to $8+6-2$.

Rule 4. When the sign − is prefixed to a vinculum containing a compound quantity, since it implies that the whole compound quantity is to be subtracted, the removal of the vinculum, in this case, requires the signs of all the terms within the vinculum to be changed, as in Subtraction. Thus, $8-(6+2)$, or $8-(+6+2)$, becomes $8-6-2$; and $8-(6-2)$ becomes $8-6+2$.

Exercise 1. Express the following quantities without the vinculum: $a+(b+c)$, $a+(b-c)$, $a-(b+c)$, and $a-(b-c)$.

2. Express the following compound quantities without the vinculum, but retaining all the terms separate: $10+(5+2)$, $10+(5-2)$, $10-(5+2)$, and $10-(5-2)$.

3. State the simple numerical value of each of the four compound quantities mentioned in the preceding exercise.

4. Simplify as much as possible the expression $(3x+2y+4z)+(x+2y+3z)$. Ans. $4x+4y+7z$.

5. The same with the expression $(3x+2y+4z)-(x+2y+3z)$. Ans. $2x+z$.

6. The same with the expression $(5a-7b+6c)+5a-10b-20c)$. Ans. $10a-17b-14c$.

7. The same with the expression $(5a-7b+6c)-(5a-10b-20c)$. Ans. $+3b+26c$.

Rule 5. In like manner, *the introduction of a vinculum*, if it is to be preceded by a positive sign, requires that all the terms enclosed shall retain their signs without alteration; but if the vinculum is to be preceded by a negative sign, all the terms enclosed must have their signs changed.

Exercise 1. Express the following compound quantities, with the last two terms of each enclosed in a vinculum preceded by the positive sign: $a+b+c$, $a+b-c$, and $a-b+c$.

2. Express each of the following compound quantities, with the the two last terms enclosed in a vinculum preceded by the negative sign: $a-b-c$, $a-b+c$, and $a+b-c$.

CHAPTER V.

MULTIPLICATION.

PROBLEM I.

To multiply two simple Quantities together.

RULE. Multiply the two co-efficients together by common Arithmetic. After the product of the co-efficients write the letters of both factors together in order; and before the product place the sign, which will always be + if the signs of the two factors are alike, and − if they are different.

Thus, the product of $+a$ by $+b$ is $+ab$; that of $-ax$ by $-by$ is $+abxy$; that of $+3xy$ by $-2ab$ is $-6abxy$; and that of $-9a^2$ by $+7b^2$ is $-63a^2b^2$.

EXERCISE 1. Multiply $+c$ by $+d$, and $-mx$ by $-ny$.

2. Multiply $+2a$ by $-3b$, and $-7x$ by $+7y$.

3. Multiply $-6x^2$ by $+5y^2$, and $-12ax^2$ by $-25by^2$.

NOTE 1. *If the same letter occur in both factors without any exponent*, instead of writing it twice, write its square. Thus, $ab \times ac = a^2bc$.

EXERCISE 4. Multiply $+4my$ by $-7ny$.

5. Multiply $-13abx$ by $-2acx$.

NOTE 2. *When two powers of the same letter occur in the two factors*, they are multiplied together by adding their exponents. Thus $b^2 \times b^3 = b^5$, and $c^6 \times c = c^7$.

EXERCISE 6. Multiply $-8a^5$ by $-9a^7$, and $+3abx^3$ by $+11acx^2$.

7. What is the product of $+17mn^4$ and $-2m^3n$?

PROBLEM II.

To multiply a compound Quantity by a simple Quantity.

RULE. Multiply every term of the compound quantity separately, and connect the products by their proper signs.

EXERCISE 1. Multiply a^2-2b^2 by $3xy$.
Ans. $3a^2xy-6b^2xy$.

2. Multiply $7a-5$ by $-8ab$, and $8x^2-9y^2$ by $+12x^3y^3$.
Answers: $-56a^2b+40ab$, and $96x^5y^3-108x^3y^5$.

3. Multiply $6ax^3-8b^2x^2-20c^3x$ by $-5mx$.
Ans. $-30amx^4+40b^2mx^3+100c^3mx^2$.

PROBLEM III.

To multiply one compound Quantity by another.

RULE. Take either of the compound quantities as the multiplicand, and multiply that successively by the several terms of the other; then add the products together.

NOTE. In multiplying, it is usual to begin at the left hand, proceeding to the right.

EXAMPLE. Multiply $a^2+2ax+x^2$ by $a^2-2ax+x^2$.

$$\begin{array}{lllll}
a^2+2ax+x^2 & & & & \\
a^2-2ax+x^2 & & & & \\
\hline
a^4+2a^3x & +a^2x^2 & & & \\
\quad -2a^3x & -4a^2x^2 & -2ax^3 & & \\
 & +a^2x^2 & +2ax^3 & +x^4 & \\
\hline
\text{Ans. } a^4 \quad * & -2a^2x^2 & * & +x^4. &
\end{array}$$

EXERCISE 1. Multiply $a+b$ by $a+b$.
Ans. $a^2+2ab+b^2$.

2. Multiply $3m-2n$ by $2m-3n$.
Ans. $6m^2-13mn+6n^2$.

3. Multiply $2x+2y$ by $2x-2y$. Ans. $4x^2-4y^2$.

4. Multiply $5ax+5ay$ by $4bx-4by$.
Ans. $20abx^2-20aby^2$.

5. Multiply $x^2+2xy+y^2$ by $x-y$.
Ans. $x^3+x^2y-xy^2-y^3$.

6. Multiply $9a^2+6ab+4b^2$ by $3a-2b$.
Ans. $27a^3-8b^3$.

7. Multiply $-4m^2+10mn-25n^2$ by $-4m^2-10mn-25n^2$. Ans. $16m^4+100m^2n^2+625n^4$.

8. Multiply $a^3-3a^2x+3ax^2-x^3$ by $a^3+3a^2x+3ax^2+x^3$. Ans. $a^6-3a^4x^2+3a^2x^4-x^6$.

9. Multiply $7abx-9cdy$ by $12acx-10bdy$.
Ans. $84a^2bcx^2-108ac^2dxy-70ab^2dxy+90bcd^2y^2$.

10. Multiply $8a^3+10a^2+12a+14$ by $4a^3-5a^2+6a-7$. Ans. $32a^6+46a^4-68a^2-98$.

Note. When compound quantities are enclosed in vincula, they are often multiplied together as if they were simple quantities, especially when they are alike.

Exercise 11. Multiply $7(a-b)$ by $15(a-b)$.
Ans. $105(a-b)^2$.

12. Multiply $-45a^2b(m+n)$ by $+36a^2c(m+n)^2$.
Ans. $-1620a^4bc(m+n)^3$.

13. Multiply $+5\frac{a+b}{a-b}$ by $-3\frac{a+b}{a-b}$. Ans. $-15\left(\frac{a+b}{a-b}\right)^2$.

14. Multiply $-13(x+y)^2$ by $-8(x+y)^3$.
Ans. $104(x+y)^5$.

Problem IV.

To find the Product of any Number of Factors, whether simple or compound.

Rule. Multiply the first by the second; then that product by the third; and so on.

Exercise 1. Multiply $2a$, $3b$, $5c$, and $8d$, continually together. Ans. $240abcd$.

2. Find the product of $+7ax$, $-5by$, and $-4cz$.
Ans. $+140abcxyz$.

3. Express in its simplest form $(-15a^2x^4)\times(-12ax^2)\times(-7)$. Ans. $-1260a^3x^6$.

4. What is the product of the three factors, a^2+b^2, $a+b$, and $a-b$? Ans. a^4-b^4.

5. What is the product of $a+b-c$, $a-b+c$, and $-a+b+c$?
Ans. $-a^3+a^2b+ab^2-b^3-2abc+a^2c+ac^2+b^2c+bc^2-c^3$.

6. Multiply together $x-3$, $x+3$, $x-5$, and $x+5$.
Ans. x^4-34x^2+225.

7. Also, $x+y$, $x-y$, x^2+xy+y^2, and x^2-xy+y^2.
Ans. x^6-y^6.

CHAPTER VI.

DIVISION.

PROBLEM I.

To divide one simple Quantity by another.

RULE. Cancel all the factors common to the dividend and divisor. Then, if all the factors of the divisor disappear, the quotient will be an integer consisting of the uncancelled factors of the dividend; but, if not, the quotient will be a fraction, having for its numerator the uncancelled factors of the dividend, and for its denominator those of the divisor.

The rule *for the sign* will be similar to that for multiplication,—viz., If the signs of the divisor and dividend are alike, that of the quotient will be $+$; but if unlike, it will be $-$.

NOTE 1. The common factors *of the co-efficients* will often be seen at a glance; but, if not, they are all cancelled at once by dividing both co-efficients by their greatest common measure, as found by Arithmetic. The common factors *among the letters* will be seen by mere inspection, when different powers of the same letter do not enter into the terms; but when the divisor and the dividend contain different powers of the same quantity, cancel that power which has the smaller exponent, and, at the same time, diminish the exponent of the higher power by that of the lower.

It is usual to write the divisor below the dividend in the form of a fraction, in order that the common factors may be seen more readily by standing in contiguity.

EXAMPLE 1. Divide $63abc^5$ by $-9ac^3$.

$$\frac{+63abc^5}{-9ac^3} = -7bc^2, \text{ Ans.}$$

EXAMPLE 2. Divide $-40mn^2x^4y^6$ by $-48n^2x^5y^2$.

$$\frac{-40mn^2x^4y^6}{-48n^2x^5y^2} = +\frac{5my^4}{6x}, \text{ Ans.}$$

EXERCISE 1. Divide $6ab$ by $2b$, $25xy$ by $20yz$, $+16mn$ by $-8m$, and $-7ab$ by $-3abc$.

2. Divide a^7 by a^4, $108ab^3c^4$ by $-9ab^5c^2$, $-7xyz$ by $+56x^2y^3z^4$, and $-78mx^3y^7$ by $-65my^6$.

Note 2. If the letters of the divisor are all cancelled, but not the co-efficient, we may give the letters of the quotient in an integral form, and the co-efficient in a fractional form; and if the co-efficient of the divisor is cancelled, and not all its letters, we may give the co-efficient of the quotient as an integer and the letters as a fraction. Thus, $3a^5 \div 4a^2 = \frac{3}{4}a^3$, and $12xy \div 3yz = 4\frac{x}{z}$.

Exercise 3. Divide $42abc^2$ by $36bc$, and $-108x^2y^3$ by $132xy$.

4. Divide $81ab^2x$ by $-27a^2bx$, and $-104my^6$ by $-8ny^3$.

Problem II.

To divide a compound Quantity by a simple Quantity.

Rule. Divide the terms of the dividend separately, and connect the quotients by their proper signs.

Exercise 1. Divide $a^3x^2 - 5abx^3 + 6ax^4$ by ax^2.
Ans. $a^2 - 5bx + 6x^2$.

2. How often is $-8xy$ contained in $-16mx^3y + 24nx^2y^2 + 56pxy^3$? Ans. $2mx^2 - 3nxy - 7py^2$.

3. Divide $30m^2n^3x^4 - 50m^3n^2x^4$ by $8m^2n^2x^2$.
Ans. $\frac{15}{4}nx^2 - \frac{25}{4}mx^2$.

4. Divide $56a^2y - 72b^2y$ by $-8aby^2$.
Ans. $-7\frac{a}{by} + 9\frac{b}{ay}$.

Problem III.

To divide one compound Quantity by another.

Rule. Arrange both the divisor and the dividend according to the powers of some one letter contained in them: then divide the first term of the dividend by the first term of the divisor, for the first term of the quotient. Multiply the whole divisor by the term thus found. Subtract the product from the corresponding terms of the dividend: bring down to the remainder as many other terms of the dividend as may be required; and repeat the operation till all the terms are brought down.

Note 1. The answers should all be proved by multiplication.

Example. Divide $a^3 - 6a^2b + 12ab^2 - 8b^3$ by $a - 2b$.

$$
\begin{array}{l}
a - 2b)a^3 - 6a^2b + 12ab^2 - 8b^3(a^2 - 4ab + 4b^2, \text{ Ans.} \\
\qquad\quad a^3 - 2a^2b \\
\hline
\qquad\qquad - 4a^2b + 12ab^2 \\
\qquad\qquad - 4a^2b + 8ab^2 \\
\hline
\qquad\qquad\qquad\quad + 4ab^2 - 8b^3 \\
\qquad\qquad\qquad\quad + 4ab^2 - 8b^3.
\end{array}
$$

Exercise 1. Divide $a^2 - 2ab + b^2$ by $a - b$. Ans. $a - b$.

2. Divide $6a^2 + 5ab - 6b^2$ by $2a + 3b$. Ans. $3a - 2b$.

3. Divide $6m^3 - 13m^2 + 6m$ by $3m - 2$. Ans. $2m^2 - 3m$.

4. Divide $am - an + bm - bn$ by $m - n$. Ans. $a + b$.

5. Divide $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$ by $x^2 - 2xy + y^2$. Ans. $x^3 - 3x^2y + 3xy^2 - y^3$.

6. Divide $6xy^3 + 6x^2y^2 - 8x^3y - 8x^4$ by $2x^2 + 2xy$. Ans. $-4x^2 + 3y^2$.

7. Divide $12a^4 - 192$ by $3a - 6$. Ans. $4a^3 + 8a^2 + 16a + 32$.

8. Divide $64z^6 - 729$ by $4z^2 - 9$. Ans. $16z^4 + 36z^2 + 81$.

9. Divide $10a^5 - 21a^4b + 27a^3b^2 - 34a^2b^3 + 11ab^4 + 5b^5$ by $2a^3 - 3a^2b + 4ab^2 - 5b^3$. Ans. $5a^2 - 3ab - b^2$.

Note 2. If there is a remainder after the process is completed, write the remainder as the numerator of a fraction, having the divisor for its denominator, and attach this to the quotient with the sign + or −. The former of these signs will usually be employed when the first term of the remainder is positive; but the latter, when negative.

Exercise 10. Divide $x^4 - 2x^2y^2 + 2y^4$ by $x^2 - y^2$.

Ans. $x^2 - y^2 + \dfrac{y^4}{x^2 - y^2}$.

11. Divide $4a^2 - 10b^2$ by $2a - 3b$.

Ans. $2a + 3b - \dfrac{b^2}{2a - 3b}$.

12. Divide $8z^4 - 12z^2 - 3z$ by $2z^3 - z^2 - 3z$.

Ans. $4z + 2 + \dfrac{2z^2 + 3z}{2z^3 - z^2 - 3z}$.

Note 3. When the negative sign is attached to the fraction, the numerator of which is a compound quantity, we must change all the signs of the numerator, according to Rule 5 of Chapter IV. See also Def. 33, Ch. I.

Exercise 13. Divide $a^3-3a^2b+2ab^2-2b^3$ by $a^2-2ab+b^2$. Ans. $a-b-\dfrac{ab^2+b^3}{a^2-2ab+b^2}$.

14. Divide $a^4+2a^3b+2a^2b^2+ab^3$ by $a^2+2ab+b^2$.

Ans. $a^2+b^2-\dfrac{ab^3+b^4}{a^2+2ab+b^2}$.

15. Divide $32a^5-240a^4b+720a^3b^2-1100a^2b^3+800ab^4-240b^5$ by $8a^3-36a^2b+54ab^2-27b^3$.

Ans. $+4a^2-12ab+9b^2-\dfrac{20a^2b^3+10ab^4-3b^5}{8a^3-36a^2b+54ab^2-27b^3}$.

Note 4. Sometimes, instead of the remainder being put down in a fractional form, the process of division is carried further, and the quotient continued in the form of a series; and, when a sufficient number of terms have been found to show the *law* of the series, as it is called, the quotient is concluded with "&c." Sometimes, also, a simple quantity is divided by a compound quantity in the same manner.

Exercise 10. Divide $1+a$ by $1-a$.
Ans. $1+2a+2a^2+2a^3+$ &c.

11. Divide $1-a$ by $1+a$.
Ans. $1-2a+2a^2-2a^3+$ &c.

12. Divide 1 by $1+x$. Ans. $1-x+x^2-x^3+$ &c.

13. Divide 1 by $1-x$. Ans. $1+x+x^2+x^3+$ &c.

14. Divide $9-18x^2$ by $9+18x^2$.
Ans. $1-4x^2+8x^4-16x^6+$ &c.

15. Divide 1 by $1-2a+a^2$.
Ans. $1+2a+3a^2+4a^3+$ &c.

CHAPTER VII.

COMMON MEASURES.

DEFINITIONS.

When one quantity is contained a whole number of times in another without leaving a remainder, the former quantity is said to measure the latter, or to be a *Measure* of it. Thus 3 is a measure of 12, and $a+b$ measures $ac+bc$.

When one quantity measures two or more quantities, it is called their *common Measure.* Thus, 3 is a common measure of 9 and 12, 2 is a common measure of all even numbers, and a^2 is a common measure of a^3 and a^2b.

A quantity may have several measures, and two or more quantities may have several common measures.

EXERCISE 1. What are the measures of 12?
2. What are the measures of 30?
3. What are the common measures of 12 and 30?
4. What are the measures of $4ab$?
5. What are the measures of $6ac$?
6. What are the common measures of $4ab$ and $6ac$?

PROBLEM.

To find the greatest common Measure of two given Quantities.

RULE. The greatest common measure of two quantities is the product of all the factors common to both; but it is found in different ways in different cases.

CASE 1. *When the quantities are both simple.* The common factors *among the letters* are seen at once by inspection; and, *in the co-efficients,* they may always be readily observed if the numbers are small; but, if large, their greatest common measure is found by the usual rule of Arithmetic. Thus, the greatest common measure of $4ab^2c$ and $6a^2bc$ is $2abc$.

EXERCISE 1. What is the greatest common measure of the quantities abc and acd?
2. What of a^3x^2y and a^2bx?
3. What of $6am^2n$ and $9a^2mn$?
4. What of $238ab^5c^2d$ and $182a^2b^2c^2d^2$?
5. What of $8925mx^4y^3$ and $12852nx^3y^4$?

Ans. $357x^3y^3$.

CASE 2. *When the quantities consist solely of compound factors.* Arrange the terms of each quantity according to the powers of some one letter. Of the two quantities, take that which commences with the higher power (or, if equal, either), and divide it by the other. If there is a remainder, use it as a divisor for the next operation, and by it divide the preceding divisor now used as a dividend. Continue this process until there is no remainder. The last divisor will be the greatest common measure.

Example 1. Find the greatest common measure of the two quantities x^3+2x^2+2x+1 and x^2+2x+1.

$$
\begin{array}{rl}
x^2+2x+1\,)\,x^3+2x^2+2x+1\,(\,x & \\
x^3+2x^2+x & \\
\hline
+x+1\,)\,x^2+2x+1\,(\,x+1 & \\
x^2+x & \\
\hline
x+1 & \\
x+1 &
\end{array}
$$

Ans. $x+1$.

Exercise 1. Find the greatest common measure of $a^3+a^2b-ab^2-b^3$ and a^2-b^2. Ans. a^2-b^2.

2. What is the greatest common measure of $5+30a+16a^3-6a^4+3a^6$ and $a^5-2a^3+5a^2-10$? Ans. a^3+5.

3. Of x^4-3x^2+2 and $x^6-3x^4+3x^2-2$?

Ans. x^2-2.

Note 1. When the sign of the first term of any divisor is negative, we may change the sign of *all* the terms if we please.

Note 2. If, in the course of the process just described, the first term of any of the divisors does not measure the first term of the corresponding dividend, divide the former by any simple factor or factors which will divide all its terms, if any such can be found. But, if none—or if, after doing so, the first term of the divisor does not yet measure the first term of the dividend—multiply the dividend by the smallest simple factor necessary to effect that object.

Example 2. Determine the greatest common measure of a^3-ab^2 and $a^2+2ab+b^2$.

$$
\begin{array}{r}
a^2+2ab+b^2\,)\,a^3 \qquad\quad -ab^2\,(\,a. \\
a^3+2a^2b+ab^2 \\
\hline
-2a^2b-2ab^2.
\end{array}
$$

Dividing the remainder (which is to be our next divisor, by $2ab$, according to Note 2, and then changing the signs of both its terms, by Note 1, it becomes $a+b$.

$$
\begin{array}{r}
a+b\,)\,a^2+2ab+b^2\,(\,a+b \\
a^2+ab \\
\hline
ab+b^2 \\
ab+b^2.
\end{array}
$$

Ans. $a+b$.

Example 3. What is the greatest common measure of $3a^2+ab-4b^2$ and $4a^2-5ab+b^2$?

Making the second quantity the dividend, we must multiply it by 3 before its first term can be measured by the first term of the divisor.

$$\begin{array}{rl}
 & 4a^2-\ \ 5ab+\ \ b^2\\
 & \qquad\qquad\quad\ 3\\
\cline{2-2}
3a^2+ab-4b^2\,) & 12a^2-15ab+\ 3b^2\,(\,4.\\
 & 12a^2+\ \ 4ab-16b^2\\
\cline{2-2}
 & \qquad -19ab+19b^2
\end{array}$$

Divide the remainder by $19b$, and change its signs (making $a-b$) for the next divisor.

$$\begin{array}{rl}
a-b\,) & 3a^2+\ ab-4b^2\,(\,3a+4b.\\
 & 3a^2-3ab\\
\cline{2-2}
 & \qquad\ 4ab-4b^2\\
 & \qquad\ 4ab-4b^2.
\end{array}$$

Ans. $a-b$.

Exercise 4. Compute the greatest common measure of $6a^2+7ax-3x^2$ and $6a^2+11ax+3x^2$. Ans. $2a+3x$.

5. The same of $12x^3+4x^2-3x-1$, and $8x^3-4x^2-2x+1$. Ans. $4x^2-1$.

6. Of x^2+3x-4 and x^3-3x^2-10x. Ans. 1.

7. Of $4x^3-8ax^2-a^2x+2a^3$, and $4a^2-x^2$. Ans. $2a-x$.

8. Of $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ and $a^3+x^3-ax^2-a^2x$. Ans. $a^2-2ax+x^2$.

Note 3. In some instances the preceding directions and all others are insufficient. In such cases, the greatest common measure can be found only by resolving the two quantities into their factors, which is sometimes a troublesome process.

Case 3. *When the quantities consist of both simple and compound factors.* Find the greatest common measure of the simple factors, by Case 1; then that of the compound factors, by Case 2. The product of the two common measures thus found will be the greatest common measure of the two given quantities.

Exercise 1. Find the greatest common measure of the two quantities $8(a^2+2ab+b^2)$ and $12\,(a^3-a^2b-ab^2+b^3)$. Ans. $4(a+b)$.

2. The same of $35x^4-35y^4$ and $63x^2-63y^2$

Ans. $7x^2-7y^2$.

CHAPTER VIII.

COMMON MULTIPLES.

DEFINITIONS.

A *Multiple* is the reverse of a measure. When one quantity measures another, the latter is said to be a multiple of the former. Thus, 20 is a multiple of 5, and ab is a multiple of a.

When one quantity is a multiple of two or more quantities, it is called their *common Multiple.* Thus, 12 is a common multiple of 1, 2, 3, 4, and 6. Thus, also, abc is a common multiple of a, b, ab, ac, and bc.

PROBLEM I.

To find the least common Multiple of two given Quantities.

RULE. Divide one of the two by their greatest common measure, and multiply the other by the quotient.

NOTE. If they have no common measure greater than 1, their least common multiple is, of course, their product.

EXERCISE 1. What is the least common multiple of ab and ac? Ans. abc.

2. Of $8x^2y$ and $20xy^2$? Ans. $40x^2y^2$.

3. Of $4x^2+12x$ and $5x+15$? Ans. $20x^2+60x$.

4. Of $2m-5$ and $2m+7$? Ans. $4m^2+4m-35$.

5. Of $4x^2-4xy+y^2$ and $4x^2-y^2$? Ans. $8x^3-4x^2y-2xy^2+y^3$.

6. Of x^3+x^2+x+1 and x^3-x^2+x-1. Ans. x^4-1.

PROBLEM II.

To find the least common Multiple of three or more given Quantities.

RULE 1. Find the least common multiple of the first and second;—then of that multiple and the third quantity;—then of the last multiple and the fourth quantity; and so on, till all the quantities are taken in.

EXAMPLE. Find the least common multiple of $16x^2y$, $20xyz$, and $25yz^2$.

Operation. L. C. M. of $16x^2y$ and $20xyz$ is $80x^2yz$.
L. C. M. of $80x^2yz$ and $25yz^2$, is $400x^2yz^2$.

RULE 2. Write the given quantities in a line one after another, omitting any one that is a factor of one of the others. Divide as many of them as possible by any prime factor which measures more than one of the given quantities. Write the several quotients, and the undivided quantities, in a new line under the former.

Proceed in the same manner with the quantities in this line, continuing the process until it appears that no two of the quotients or undivided quantities have a common measure.

Then the continued product of the several divisors and of the quantities in the last line will be the least common multiple required.

EXAMPLE. What is the least common multiple of the quantities $2a$, $3b$, $4c$, $6ab$, $12ac$, and $20bc$?

Operation. We first omit $2a$, because it is a factor of $6ab$; then $3b$, for the same reason; and $4c$, because it is a factor of $12ac$. Then, by the rule,

$$\begin{array}{r|l} 2 & 6ab,\ 12ac,\ 20bc \\ \hline 3 & 3ab,\ \ 6ac,\ 10bc \\ \hline 2 & ab,\ \ 2ac,\ 10bc \\ \hline a & ab,\ \ \ ac,\ \ 5bc \\ \hline & b,\ \ \ \ c,\ \ 5bc \\ \hline & \qquad\quad 5bc. \end{array}$$

$2\times3\times2\times a\times5bc=60abc$, Ans.

EXERCISE 1. What is the least common multiple of 12, 15, and 20? Ans. 60.

2. Of xy, yz, and xz? Ans. xyz.

3. Of $3a$, $4b$, $5c$, $8a^2b$, $12ac^2$, and $15b^2c$?
Ans. $120a^2b^2c^2$.

4. Of $7x$, $9y$, and $11z$? Ans. $693xyz$.

5. Of $14x^2$, $21xy$, $6yz$, $20xz$, and $15z^2$?
Ans. $420x^2yz^2$.

6. Of x^2-y^2, $x^2+2xy+y^2$, and $x^2-2xy+y^2$?
Ans. $x^4-2x^2y^2+y^4$.

7. Of $x-y$, $x+2$, x^2+4, and x^2-4? Ans. x^4-16.

CHAPTER IX.

FRACTIONS.

NOTE. The *Definitions* of the terms usually employed in fractions are the same as in common Arithmetic. It is therefore needless to repeat them here.

PROBLEM I.

To reduce an integral Quantity to a Fraction having a given Denominator.

RULE. Multiply the whole number by the given denominator, for the numerator of the fraction, and write under it the given denominator.

EXERCISE 1. Reduce 7 to a fraction having its denominator 4. Ans. $\frac{28}{4}$.

2. Reduce a to a fraction whose denominator shall be b. Ans. $\frac{ab}{b}$.

3. Reduce $5ax$ to a fraction having 1 for its denominator.

4. Reduce $7xy^2$ to a fraction the denominator of which is given,—viz., $5xyz$. Ans. $\frac{35x^2y^3z}{5xyz}$.

5. Reduce $a+x$ to a fraction having its denominator $a-x$. Ans. $\frac{a^2-x^2}{a-x}$.

PROBLEM II.

To reduce a mixed Quantity to the Form of a Fraction.

RULE. Multiply the whole number by the denominator of the given fraction, and add its numerator to the product, if the sign of the fraction is positive; but subtract it, if negative. Write the result for the numerator of the required fraction, and, for its denominator, that of the *given fraction.*

Exercises. Reduce the following mixed numbers to fractional forms:

1. $a+\frac{b^2}{c}$. Ans. $\frac{ac+b^2}{c}$.

2. $x-\frac{y^2}{2z}$. Ans. $\frac{2xz-y^2}{2z}$.

3. $7x^2+\frac{3y^5}{5x^3}$. Ans. $\frac{35x^5+3y^5}{5x^3}$.

4. $3a-\frac{4ax+a^2}{4x}$. Ans. $\frac{8ax-a^2}{4x}$.

5. $a^2-b^2+\frac{b^4}{a^2+b^2}$. Ans. $\frac{a^4}{a^2+b^2}$.

6. $3x-4y-\frac{5xy+12y^2}{4x-3y}$. Ans. $\frac{12x^2-30xy}{4x-3y}$.

7. $8n-3-\frac{8mn-6m}{m-1}$. Ans. $\frac{3m-8n+3}{m-1}$.

8. $x-y-\frac{x^3+y^3}{x^2+xy+y^2}$. Ans. $-\frac{2y^3}{x^2+xy+y^2}$.

9. $a^4+ax^3+\frac{ax^6-x^7}{a^3-x^3}$. Ans. $\frac{a^7-x^7}{a^3-x^3}$.

Problem III.

To reduce an improper Fraction to the Form of an integral, or of a mixed, Quantity.

Rule. Divide the numerator by the denominator.

Exercises. Change the following fractions into mixed quantities:

1. $\frac{ac+b^2}{c}$. Ans. $a+\frac{b^2}{c}$.

2. $\frac{2xz-y^2}{2z}$. Ans. $x-\frac{y^2}{2z}$.

3. $\frac{35x^5+3y^5}{5x^3}$. Ans. $7x^2+\frac{3y^5}{5x^3}$.

4. $\frac{8ax-a^2}{4x}$. Ans. $2a-\frac{a^2}{4x}$.

5. $\frac{a^4}{a^2+b^2}$. Ans. $a^2-b^2+\frac{b^4}{a^2+b^2}$.

6. $\frac{12x^2-30xy}{4x-3y}$. Ans. $3x-\frac{21xy}{4x-3y}$.

7. $\frac{3m-8n+3}{m-1}$. Ans. $3-\frac{8n-6}{m-1}$.

8. $\frac{2x^3-2y^3}{x^2+xy+y^2}$. Ans. $2x-2y$.

9. $\frac{a^7-x^7}{a^3-x^3}$. Ans. $a^4+ax^3+\frac{ax^6-x^7}{a^3-x^3}$.

Note. A fraction is not increased if both its terms are multiplied by the same quantity; neither is it diminished if both are divided by the same quantity.

Definition. Any quantity by which both terms are multiplied or divided is called their *common Multiplier or common Divisor*. The former is placed to the right of the fraction, as $\frac{3}{4}(2$; the latter to the left, as $2)\frac{6}{8}$. Thus $\frac{3}{4}(2=\frac{6}{8}$, and $2)\frac{6}{8}=\frac{3}{4}$.

Problem IV.

To change one Fraction into another of equal Value having a given greater Denominator.[*]

Rule. Divide the greater denominator by the smaller for the common multiplier. Place that to the right of the given fraction, and multiply the numerator by it.

Example. Change $\frac{a}{2b}$ into a fraction having its denominator $6bc$. $\frac{a}{2b}(3c=\frac{3ac}{6bc}$. Ans.

Exercise 1. Change $\frac{x}{y}$ into a fraction having its denominator y^2. Ans. $\frac{xy}{y^2}$.

2. Reduce $\frac{2a}{3}$ to a fraction having its denominator $12ab$. Ans. $\frac{8a^2b}{12ab}$.

3. Reduce $\frac{3m}{4n}$ to the denominator $20m^2n^2$.

Ans. $\frac{15m^3n}{20m^2n^2}$

---

[*] It is not absolutely necessary that the greater denominator be a multiple of the smaller. When not so, we are merely driven into the form of a complex *fraction*.

4. Convert $\frac{7}{a-b}$ into a fraction having its denominator $2a^2-2ab$. Ans. $\frac{14a}{2a^2-2ab}$.

5. Alter the form of the fraction $\frac{5}{a+x}$, so that its denominator may become a^2-x^2. Ans. $\frac{5a-5x}{a^2-x^2}$.

6. What fraction, with the denominator $4a^2-12a+9$, is equivalent to the fraction $\frac{2a+3}{2a-3}$? Ans. $\frac{4a^2-9}{4a^2-12a+9}$.

PROBLEM V.

To reduce a Fraction to its lowest Terms.

RULE. Cancel all the factors common to the numerator and denominator; or, in other words, divide both by their greatest common measure.

EXAMPLE. Reduce $\frac{3ac}{6bc}$ to its lowest terms.

$$3c\,)\frac{3ac}{6bc}=\frac{a}{2b}.\ \text{Ans.}$$

EXERCISES. Reduce the following fractions to their lowest terms:

1. $\frac{xy}{y^2}$. Ans. $\frac{x}{y}$.
2. $\frac{15m^3n}{20m^2n^2}$. Ans. $\frac{3m}{4n}$.
3. $\frac{959a^5b^3c}{685a^2b^4c^3}$. Ans. $\frac{7a^3}{5bc^2}$.
4. $\frac{5a-5x}{a^2-x^2}$. Ans. $\frac{5}{a+x}$.
5. $\frac{m^2+3m+2}{m^3+3m^2+4m+2}$. Ans. $\frac{m+2}{m^2+2m+2}$.
6. $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$. Ans. $\frac{a^2+x^2}{a-x}$.
7. $\frac{a^5-2a^4-15a^3}{a^4-8a^3+15a^2}$. Ans. $\frac{a^4+3a^3}{a^3-3a^2}$.

8. $\frac{114(a+b)^3}{152(a+b)^2}$. Ans. $\frac{3(a+b)}{4}$.

9. $\frac{20(a^2-2ab+b^2)}{35(a^3-2a^2b+2ab^2-b^3)}$. Ans. $\frac{4(a-b)}{7(a^2-ab+b^2)}$.

Problem VI.

To reduce Fractions having different Denominators to equivalent Fractions having the same Denominator.

Rule. Take any common multiple of all the denominators for the common denominator, and reduce all the fractions to that, by Problem IV.

Note. The common multiple most easily found is the product of all the denominators; and, in that case, every numerator is multiplied by all the denominators but its own, for the new numerators. But much work is saved, in many operations, by using the *least* common multiple, found by Problem I or II of the last Chapter.

Example. Reduce $\frac{a}{4xy}$, $\frac{b}{6y^2}$, and $\frac{c}{10yz}$, to fractions having a common denominator.

$\frac{a}{4xy}(15yz = \frac{15ayz}{60xy^2z}$.

$\frac{b}{6y^2}(10xz = \frac{10bxz}{60xy^2z}$.

$\frac{c}{10yz}(6xy = \frac{6cxy}{60xy^2z}$.

Exercise 1. Reduce $\frac{a}{x}$, $\frac{b}{y}$, and $\frac{c}{z}$ to the same denominator.

Ans. $\frac{ayz}{xyz}$, $\frac{bxz}{xyz}$, and $\frac{cxy}{xyz}$.

2. Reduce $\frac{x}{4ab}$ and $\frac{y}{6ac}$ to a common denominator.

Ans. $\frac{3cx}{12abc}$, and $\frac{2by}{12abc}$.

3. Reduce $\frac{5}{6a}$, $\frac{3}{4b}$, and $\frac{4}{5c}$ to fractions having a common denominator.

Ans. $\frac{50bc}{60abc}$, $\frac{45ac}{60abc}$, and $\frac{48ab}{60abc}$.

4. The same with $\frac{2a}{15mn}$, $\frac{3b}{8mp}$, and $\frac{5c}{12np}$.

Ans. $\frac{16ap}{120mnp}$, $\frac{45bn}{120mnp}$, and $\frac{50cm}{120mnp}$.

5. The same with $\frac{a+x}{a-x}$ and $\frac{a-x}{a+x}$.

Ans. $\frac{a^2+2ax+x^2}{a^2-x^2}$ and $\frac{a^2-2ax+x^2}{a^2-x^2}$.

6. The same with $\frac{1}{a^2+2ax+x^2}$, and $\frac{1}{a^2-x^2}$, and $\frac{1}{a^2-2ax+x^2}$.

Ans. The numerators will be $a^2-2ax+x^2$, a^2-x^2, and $a^2+2ax+x^2$; and the common denominator $a^4-2a^2x^2+x^4$.

Problem VII.

To add Fractions together.

Rule. *If they have all the same denominator*, add their numerators, and under their sum, as a numerator, write the common denominator.

But *if they have not the same denominator*, reduce them to a common denominator, and then proceed according to the foregoing direction.

Exercise 1. Add together $\frac{3}{a}$, $\frac{5}{a}$, and $\frac{7}{a}$. Ans. $\frac{15}{a}$.

2. $\frac{3a+b}{5x}$, $\frac{2a-3b}{5x}$, and $\frac{-a+2b}{5x}$. Ans. $\frac{4a}{5x}$.

3. $\frac{3a}{2}$, and $\frac{2a}{3}$. Ans. $\frac{13a}{6}$.

4. $\frac{x}{4}$, $\frac{x}{5}$, $\frac{x}{6}$, and $\frac{x}{10}$. Ans. $\frac{43x}{60}$.

5. $\frac{a}{b}$ and $\frac{c}{d}$. Ans. $\frac{ad+bc}{bd}$.

6. $\frac{7m+12n}{8}$ and $\frac{12m-7n}{10}$. Ans. $\frac{83m+32n}{40}$.

7. $\frac{2a}{3b}$, $\frac{3b}{4c}$, and $\frac{4c}{5a}$. Ans. $\frac{40a^2c+45ab^2+48bc^2}{60abc}$.

8. $\frac{a}{x^2y}$ and $\frac{a}{xy^2}$. Ans. $\frac{ax+ay}{x^2y^2}$.

9. $\frac{3}{4m^2}$, $\frac{4}{5mn}$, $\frac{5}{6n^2}$. Ans. $\frac{50m^2+48mn+45n^2}{60m^2n^2}$.

10. $\frac{m+n}{m-n}$ and $\frac{m-n}{m+n}$. Ans. $\frac{2m^2+2n^2}{m^2-n^2}$.

11. $\frac{1}{y^2+2yz+z^2}$, $\frac{1}{y^2-z^2}$, and $\frac{1}{y^2-2yz+z^2}$.

Ans. $\frac{3y^2+z^2}{y^4-2y^2z^2+z^4}$.

Note 1. *If any of the fractions to be added has the negative sign*, consider that sign as attached to its numerator if simple; but if compound, change all the signs of the numerator, or suppose them changed, in adding the numerators. In the latter case, however, a negative sign attached to a fraction must be carefully distinguished from a negative sign attached to the first term of its numerator. (See Chap. I, 32, and Chap. IV, Rules 1, 2, and 3.)

Exercise 12. What is the sum of $+\frac{c}{15ab}$, $+\frac{a}{21bc}$, and $-\frac{b}{28ac}$? Ans. $\frac{20a^2-15b^2+28c^2}{420abc}$.

13. $\frac{a+2b}{x}$, $-\frac{a+b}{x}$, and $\frac{2a+b}{x}$? Ans. $\frac{2a+2b}{x}$,

14. $\frac{a^2+2ax+x^2}{a-x}$ and $-\frac{a^2-2ax+x^2}{a+x}$. Ans. $\frac{6a^2x+2x^3}{a^2-x^2}$.

15. $\frac{2a}{a^2+2ax+x^2}$ and $\frac{2x}{a^2-x^2}$.

Ans. $\frac{2a^2+2x^2}{a^3+a^2x-ax^2-x^3}$.

Note 2. The fraction expressing the sum, if not in its lowest terms, must be reduced to them, or, if possible, to a whole number.

Exercise 16. Add $\frac{a+b}{2}$ to $\frac{a-b}{2}$. Ans. a.

17. Add $\frac{4ax}{a^2-x^2}$ to $\frac{a-x}{a+x}$. Ans. $\frac{a+x}{a-x}$.

Problem VIII.

To subtract one Fraction from another.

Rule. Proceed as in addition of fractions, subtracting instead of adding.

Exercise 1. From $\frac{38mx}{ay}$ take $\frac{15mx}{ay}$. Ans. $\frac{23mx}{ay}$.

2. Take $\frac{2a^3-3a^2}{13b}$ from $\frac{5a^3+7a^2}{13b}$. Ans. $\frac{3a^3+10a^2}{13b}$.

3. $\frac{5m}{6}-\frac{3m}{4}$. Ans. $\frac{m}{12}$.

4. $\frac{a}{x}-\frac{b}{y}$. Ans. $\frac{ay-bx}{xy}$.

5. $\frac{10a}{21}-\frac{9b}{14}$. Ans. $\frac{20a-27b}{42}$.

6. $\frac{9x-16}{4}-\frac{6x-20}{5}$. Ans. $\frac{21x}{20}$.

7. $\frac{m}{16a^2b}-\frac{n}{24ab^2}$. Ans. $\frac{3bm-2an}{48a^2b^2}$.

8. $\frac{5}{102yz}-\frac{4}{85xy}$. Ans. $\frac{25x-24z}{510xyz}$.

9. $\frac{a+x}{a-x}-\frac{a-x}{a+x}$. Ans. $\frac{4ax}{a^2-x^2}$.

10. $\frac{1}{2a-2b}-\frac{1}{2a+2b}$. Ans. $\frac{b}{a^2-b^2}$.

11. $\frac{2a}{a^2+2ax+x^2}-\frac{2x}{a^2-x^2}$. Ans. $\frac{2a^2-4ax-2x^2}{a^3+a^2x-ax^2-x^3}$.

12. $\frac{a+b}{2}-\frac{a-b}{2}$. Ans. b.

Problem IX.

To Multiply one Fraction by another, or to find the Product of any number of Fractions.

Rule. Multiply all the numerators together for the numerator of the product, and all the denominators for the denominator of the product.

The sign follows the same rule as in the case of integral quantities.

Exercise 1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$. Ans. $\frac{ac}{bd}$.

2. Multiply $+\frac{2x}{7y}$ by $+\frac{3x^2}{5y^2}$. Ans. $+\frac{6x^3}{35y^3}$.

3. Multiply $+\frac{27a^2x^2}{35b^4y}$ by $-\frac{6a^3x}{11by^2}$. Ans. $-\frac{162a^5x^3}{385b^5y^3}$.

4. What is the product of $-\frac{3a}{x}$, $+\frac{4b}{5y}$, and $-\frac{6c^2}{7xy}$?

Ans. $-\frac{72abc^2}{35x^2y^2}$.

5. What is the product of $\frac{2a+x}{2a-x}$, and $\frac{a+2x}{a-2x}$?

Ans. $\frac{2a^2+5ax+2x^2}{2a^2-5ax+2x^2}$.

NOTE 1. The same factor or factors occurring in any numerator, and also in a denominator, may be cancelled in both.

EXAMPLE. Express in its simplest form, $\frac{3a}{7b}\times\frac{5b}{9c}\times\frac{28x}{15y}$.

$$\frac{3a}{7b}\times\frac{5b}{9c}\times\frac{28x}{15y}=\frac{a}{1}\times\frac{1}{3c}\times\frac{4x}{3y}=\frac{4ax}{9cy}.$$

EXERCISES. Find the respective products of the following quantities.

6. $\frac{5a}{4x}$ and $\frac{12b}{13x}$. Ans. $\frac{15ab}{13x^2}$.

7. $-\frac{9a}{10b}$ and $+\frac{2c}{3d}$. Ans. $-\frac{3ac}{5bd}$.

8. $-\frac{a^2}{b}$ and $-\frac{b^2}{a}$. Ans. $+ab$.

9. $+\frac{21mx}{22n^2}$ and $-\frac{7nx}{9m^2}$. Ans. $-\frac{49x^2}{66mn}$.

10. $\frac{a}{b}$, $\frac{b}{c}$, and $\frac{c}{d}$. Ans. $\frac{a}{d}$.

11. $+\frac{14v}{9x}$, $-\frac{5x}{7y}$, and $+\frac{27y}{20z}$. Ans. $-\frac{3v}{2z}$.

12. $\frac{2x-3y}{2x+3y}$ and $\frac{x+y}{4x-6y}$. Ans. $\frac{x+y}{4x+6y}$.

13. $\frac{a^2-b^2}{a^2+b^2}$ and $\frac{a^3+ab^2}{a^2-2ab+b^2}$. Ans. $\frac{a^2+ab}{a-b}$.

NOTE 2. *When the product of integral quantities and fractions is required,* the former must be multiplied into the numerators of the latter, or previously made into fractions having their denominator 1.

Example. What is the product of the quantities $3x^2$, $\frac{4a}{x}$, and $\frac{b}{12y}$?

$$\frac{3x^2}{1}\times\frac{4a}{x}\times\frac{b}{12y}=\frac{x}{1}\times\frac{a}{1}\times\frac{b}{y}=\frac{abx}{y}.$$

Exercise 14. Multiply together $\frac{2m^2}{33n}$, $\frac{5n^2}{6m}$, and $18mn$.

Ans. $\frac{10m^2n^2}{11}$.

15. $-\frac{3cx}{4by}$, $+20bz$, and $-\frac{6ay}{5cz}$. Ans. $+18a$.

Note 3. *To multiply a fraction by an integer equal to the denominator of the fraction*, we merely remove the denominator. The numerator, taken as an integer, is the answer.

Exercise 16. Multiply $\frac{3}{4}$ by 4 Ans. 3.

17. Multiply $\frac{37ay}{14bx}$ by $14bx$. Ans. $37ay$.

Note 4. *To multiply a fraction by a whole number, when that number is a multiple of the denominator*,—divide the whole number by the denominator, and multiply the numerator by the quotient: the resulting integer is the answer.

Example. Multiply $\frac{7a}{3x}$ by $12xy$.

$$12xy \div 3x = 4y.$$
$$7a \times 4y = 28ay, \text{ Ans.}$$

Exercise 18. Multiply $\frac{3}{4}$ by 12. Ans. 9.

19. Multiply $\frac{3ax}{5b}$ by $35ab$. Ans. $21a^2x$.

20. Multiply $\frac{a}{3}+\frac{b}{4}-\frac{c}{6}$ by 12. Ans. $4a+3b-2c$.

21. Multiply $\frac{a+x}{a-x}$ by a^2-x^2. Ans. $(a+x)^2$.

Note 5. Mixed quantities may be multiplied either by first reducing them to a fractional form, or as compound quantities.

Exercise 22. Multiply $a+\frac{b}{c}$ into $a-\frac{b}{c}$.

Ans. $\frac{a^2c^2-b^2}{c^2}$ or $a^2-\frac{b^2}{c^2}$.

PROBLEM X.

To divide one Fraction by another.

RULE. Invert the divisor, and multiply by it when so inverted; or, in other words, multiply the dividend by the reciprocal of the divisor.

The sign follows the same rule as in division of integers.

EXAMPLE. Divide $\frac{108a^3b^2c}{7xyz}$ by $\frac{12a^2b}{11z}$.

$$\frac{108a^3b^2c}{7xyz} \div \frac{12a^2b}{11z} = \frac{108a^3b^2c}{7xyz} \times \frac{11z}{12a^2b} = \frac{99abc}{7xy}.$$

NOTE 1. *If one of the two given quantities—viz., either the divisor or the dividend—is an integer, the other being a fraction,* we proceed in the same manner, first changing the integer into a fraction by giving it the denominator 1.

EXERCISE 1. Divide $\frac{x}{y}$ by $\frac{b}{a}$. Ans. $\frac{ax}{by}$.

2. Divide $\frac{x}{y}$ by $\frac{x}{z}$. Ans. $\frac{z}{y}$.

3. Divide $\frac{15ab}{13x^2}$ by $\frac{5a}{4x}$. Ans. $\frac{12b}{13x}$.

4. Divide $+\frac{60a^3}{7b}$ by $-9ab$. Ans. $-\frac{20a^2}{21b^2}$.

5. Divide $-9ab$ by $+\frac{60a^3}{7b}$. Ans. $-\frac{21b^2}{20a^2}$.

6. Divide $-\frac{3ac}{5bd}$ by $-\frac{9a}{10b}$. Ans. $+\frac{2c}{3d}$.

7. Divide $\frac{a^2+ab}{a-b}$ by $\frac{a^3+ab^2}{a^2-2ab+b^2}$. Ans. $\frac{a^2-b^2}{a^2+b^2}$.

8. Divide $\frac{x+y}{4x+6y}$ by $\frac{2x-3y}{2x+3y}$. Ans. $\frac{x+y}{4x-6y}$.

NOTE 2. *When either the divisor or the dividend is a mixed quantity,* it may either be reduced to a fraction or be treated as a compound quantity.

EXERCISE 9. Divide $x-\frac{y}{a}$ by $\frac{x}{a}$.

Ans. $\frac{ax-y}{x}$ or $a-\frac{y}{x}$.

10. Divide $a^2-\frac{b^2}{c^2}$ by $a+\frac{b}{c}$. Ans. $\frac{ac-b}{c}$ or $a-\frac{b}{c}$.

CHAPTER X.

INVOLUTION.

Involution is the method of raising any given quantity to any given power.

PROBLEM I.

To find any proposed Power of a simple Quantity.

RULE. Multiply the exponent of every *literal factor* by the exponent of the proposed power, and to the result prefix the same power of the given *co-efficient*, as found by Arithmetic.

When the *sign* of the given quantity is +, that of every power will be +; and when the sign of the given quantity is −, that of its even powers will be +, and of its odd powers −.

NOTE. It will be recollected, that when no exponent is expressed, it is understood to be 1.

EXERCISE 1. What is the cube, or third power, of a^2?
Ans. a^6.

2. What is the square, or second power, of $7a^5b^3$?
Ans. $49a^{10}b^6$.

3. Find the fourth power of $-5xy^4$.
Ans. $+625x^4y^{16}$.

4. Compute the seventh power of $-2ab^2c^3$.
Ans. $-128a^7b^{14}c^{21}$.

PROBLEM II.

To find any proposed Power of a compound Quantity.

RULE. Raise it to that power by actual multiplication. The number of factors will, of course, be the same as the exponent of the proposed power (See Chap. I., Def. 20); and, consequently, the number of operations will be one fewer.

EXERCISE 1. Find the square of $a+b$.
Ans. $a^2+2ab+b^2$.

2. The cube of $a-b$. Ans. $a^3-3a^2b+3ab^2-b^3$.

3. The second power of $7x-9y$.
Ans. $49x^2-126xy+81y^2$.

4. The fourth power of $5x-1$.
Ans. $625x^4-500x^3+150x^2-20x+1$.

5. The fifth power of $2m-3n$.
Ans. $32m^5-240m^4n+720m^3n^2-1080m^2n^3+810mn^4-243n^5$.

6. The seventh power of $x-y$.
Ans. $x^7-7x^6y+21x^5y^2-35x^4y^3+35x^3y^4-21x^2y^5+7xy^6-y^7$.

7. The second power of $x-2y+3z$.
Ans. $x^2-4xy+6xz+4y^2-12yz+9z^2$.

8. The third power of $4a^2+5a$.
Ans. $64a^6+240a^5+300a^4+125a^3$.

9. The square of $1+\sqrt{x}$. Ans. $1+2\sqrt{x}+x$.

10. The cube of $x-\frac{2}{3}$. Ans. $x^3-2x^2+\frac{4}{3}x-\frac{8}{27}$.

11. The square of $\frac{a}{b}+\frac{b}{c}$. Ans. $\frac{a^2}{b^2}+2\frac{a}{c}+\frac{b^2}{c^2}$.

PROBLEM III.

To raise any given Fraction to any proposed Power.

RULE. Raise its numerator and denominator separately to that power.

EXERCISE 1. What is the fifth power of $\frac{2ab^2}{3y^3}$.

Ans. $\frac{32a^5b^{10}}{243y^{15}}$.

2. What is the square of $\frac{a+5}{a-3}$?

Ans. $\frac{a^2+10a+25}{a^2-6a+9}$.

CHAPTER XI.

EVOLUTION.

Evolution is the reverse of Involution. It is the method of finding any proposed root of any given quantity.

PROBLEM I.

To find any proposed Root of any given simple Quantity.

RULE. Divide the exponent of every *literal factor* in the given quantity, by the exponent of the proposed root, and prefix the proposed root of the given *co-efficient*, as found by Arithmetic.

When the sign of the given quantity is –, the extraction of any even root is impossible, and any odd root will have the sign –. *When the sign of the given quantity is* +, the sign of every odd root is +, and the sign of every even root either + or –.

NOTE. The results may be *proved* by involution.

EXERCISE 1. What are the square roots of $+a^6$ and $-a^6$?

2. What are the cube roots of $+a^6$ and $-a^6$?

3. Find the square root of $9a^2$. Ans. $\pm 3a$.

4. The cube root of $8a^3$. Ans. $2a$.

5. The fourth root of $81x^{12}y^8$. Ans. $\pm 3x^3y^2$.

6. The third root of $-125x^{12}y^6$. Ans. $-5x^4y^2$.

7. The sixth root of $-6a^4$.

8. The cube root of $343a^2$. Ans. $7a^{\frac{2}{3}}$.

9. The square root of $5{\cdot}4756 \times a^{10}$. Ans. $2{\cdot}34 \times a^5$.

PROBLEM II.

To find the Square Root of any given compound Quantity.

RULE. 1. Arrange the terms of the given quantity according to the powers of one of the letters. Find the square root of the first term by the preceding problem, and place it as the first term of the required root.

2. Place the square of the root thus found under the first term of the given quantity, to which, of course, it will be equal. Bring down the two next terms of the given quantity for a dividend, and twice the first term of the root for a divisor.

3. Divide the first term of the said dividend by the first term of the divisor. The quotient will be the second term of the root, to which it must be annexed with its proper sign, as also to the divisor.

4. Multiply the divisor, thus enlarged, by the second term of the root, and subtract the product from the last dividend.

5. To the remainder bring down two more terms of the given quantity, and use that as a new dividend. The new divisor will be the double of that part of the root already found.

6. Proceed with the new divisor and dividend as with the previous; and so on until there is no remainder, and no more terms to be brought down, or as far as may be thought necessary if the remainder do not disappear.

The whole process is exactly similar to that in common Arithmetic.

Example. Extract the square root of $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

$$\begin{array}{rl}
 & a^4-4a^3b+6a^2b^2-4ab^3+b^4\,(\;a^2-2ab+b^2. \\
 & a^4 \\
 & \overline{} \\
2a^2-2ab\,) & -4a^3b+6a^2b^2 \\
 & -4a^3b+4a^2b^2 \\
 & \overline{} \\
2a^2-4ab+b^2\,) & 2a^2b^2-4ab^3+b^4 \\
 & 2a^2b^2-4ab^3+b^4
\end{array}$$

The required root is, $a^2-2ab+b^2$.

Exercise 1. Find the square root of $x^2+2xy+y^2$. Ans. $x+y$.

2. Of $9m^2-12mn+4n^2$. Ans. $3m-2n$.

3. Of $x^4-2x^2y^2+y^4+2x^2z^2-2y^2z^2+z^4$. Ans. $x^2-y^2+z^2$.

4. Of $a^6-4a^5-2a^4+12a^3+9a^2$. Ans. a^3-2a^2-3a.

5. Of $4x^2-12xy+9y^2+16xz-24yz+16z^2$. Ans. $2x-3y+4z$.

6. Of $25a^2+10ab-2ac+b^2-\frac{2}{5}bc+\frac{1}{25}c^2$. Ans. $5a+b-\frac{1}{5}c$.

7. Of $9x^6-12x^5+10x^4-28x^3+17x^2-8x+16$. Ans. $3x^3-2x^2+x-4$.

Note 1. In each of the preceding answers the signs may be changed (provided we change all of them), and the result will be another answer equally correct.

Note 2. The extraction of other roots than the second *is a complicated* process, and seldom required.

CHAPTER XII.

PROPORTION.

DEFINITIONS.

1. When the first of four quantities contains the second as often as the third contains the fourth, the four quantities are said to be *Proportional.* Thus, since $\frac{16}{4}=\frac{12}{3}$, we say, $16 : 4 :: 12 : 3$; and if $\frac{a}{b}=\frac{c}{d}$, then $a : b :: c : d$.

Note. When we speak of the first containing the second as often as the third contains the fourth, we mean that the quotients are equal, whether they be integers, fractions, or mixed numbers, or even if they be roots or other expressions not convertible at all into definite numbers. Thus, $3 : 4 :: 9 : 12$, because $\frac{3}{4}=\frac{9}{12}$. Also $2 : \sqrt{20} :: 3 : \sqrt{45}$, although we cannot express either $2 \div \sqrt{20}$ or $3 \div \sqrt{45}$ exactly, in numbers, at all.

2. When four quantities are proportional, the first is said to have to the second the same *Ratio* which the third has to the fourth. A proportion, therefore, expressing the equality of two ratios, is sometimes written thus, $16 : 4 = 12 : 3$, which is read thus,—the ratio of 16 to 4 is equal to the ratio of 12 to 3.

3. The first term of a ratio is called its *Antecedent,* and the second its *Consequent.* Thus, in the proportion $16 : 4 :: 12 : 3$, the antecedents are 16 and 12, and the consequents 4 and 3.

4. The first and fourth terms of a proportion are called the *Extremes;* and the second and third, the *Means.*

5. The fourth term is said to be a *Fourth Proportional* to the other three.

6. When the first of three given quantities has the same ratio to the second, which the second has to the third, the last is called a *Third Proportional,* and the second a *Mean Proportional* to the other two.

THEOREM I.

In any proportion the product of the extreme terms is equal to the product of the means.

Thus, since $3 : 4 :: 9 : 12$, we have $3\times12=4\times9$; and, generally, if $a : b :: c : d$, then $ad=bc$.

NOTE. This theorem forms a good criterion for determining if four given quantities are proportional.

EXERCISE 1. Are the numbers 51, 85, 39, and 65 proportional?

2. Is it correct to say,—As $28 : 42 :: 46 : 70$?

3. Are the four quantities $8a$, $12b\sqrt{x}$, $10\frac{a}{b}$, and $15\sqrt{x}$ in proportion?

THEOREM II.

If three quantities are proportional, the product of the extremes is equal to the square of the mean.

EXERCISE 1. Are the three numbers, 336, 420, and 525, proportionals?

2. Is $36a^2x^2$ a mean proportional between $54ax^3$ and $24a^3x$?

THEOREM III.

If the product of two quantities be equal to the product of two others, the four are proportionals, making the factors of the one product the means, and the factors of the other the extremes.

EXERCISE. Since $8\times15=10\times12$, show how many proportions can be stated among these four numbers.

THEOREM IV.

If four quantities are proportionals, they are also proportionals when stated inversely.

Thus, if $a : b :: c : d$, then $b : a :: d : c$.

EXERCISE. Take the proportion, $75 : 100 :: 57 : 76$, and state it inversely.

THEOREM V.

If four quantities are proportionals, they are also proportionals when stated alternately.

Thus, if $a : b :: c : d$, then $a : c :: b : d$.

EXERCISE. Since $20ax : 15bx :: 12ay : 9by$, state the proportion alternately.

THEOREM VI.

When four quantities are proportionals, we may multiply or divide either of the extremes by any number, provided that we multiply or divide one of the means by the same number, and the proportion will still hold good.

Thus, if $a : b :: c : d$, then $3a : 3b :: c : d$, and $5a : b :: 5c : d$. Also, $\frac{a}{2} : b :: \frac{c}{2} : d$, &c., &c. In general $ma : mb :: nc : nd$, and $ma : nb :: mc : nd$, m and n being any numbers, integral or fractional.

EXERCISE 1. Change the first and second terms of the proportion $40 : 60 :: x : y$ into smaller numbers.

2. Change the proportion $\frac{p}{2} : x :: \frac{q}{2} : y$, into another containing only integral terms.

3. The same with the proportion $\frac{a}{3} : \frac{b}{2} :: \frac{c}{6} : \frac{d}{5}$.

Ans. $2a : 5b :: c : 2d$.

4. Change the proportion $(a+x)^2 : a^2-x^2 :: 15m^2 : 25mn$ into one with simpler terms.

Ans. $a+x : a-x :: 3m : 5n$.

THEOREM VII.

If four quantities be proportionals, like powers or roots of those quantities will also be proportionals.

Thus, if $a : b :: c : d$, then $a^3 : b^3 :: c^3 :: d^3$, and $\sqrt{a} : \sqrt{b} :: \sqrt{c} : \sqrt{d}$, &c., &c.

Exercise 1. Square all the terms of the proportion $25 : 35 :: 15 : 21$, and try if the results are proportional, by Theorem I.

2. Change the proportion $9a^2 : 25b^2 :: 4c^4 : 36d^4$, into one composed of simpler terms.

PROBLEM I.

To find any Term of a Proportion, the other three Terms being given.

Rule. If the required term is one of the means, divide the product of the two extremes by the other mean. But if the required term is one of the extremes, divide the product of the two means by the other extreme.

Exercise 1. Supply the blank term in each of the following proportions:—

$4 : 5 :: 12 : \text{——}.$	Ans. 15.
$20 : 15 :: \text{——} : 39.$	52.
$102 : \text{——} :: 78 : 130.$	170.
$\text{——} : 84 :: 70 : 98.$	60.
$13a : 15a :: 13x : \text{——}.$	$15x.$
$2a : 3\sqrt{b} :: \text{——} : 21a.$	$14.\frac{a^2}{\sqrt{b}}.$
$a + x : \text{——} :: a : b.$	$b + \frac{b}{a}.x.$

Exercise 2. Two quantities are to each other as 3 to 4. If we call the former of the two $3x$, what must we call the latter; and if we call the latter $60x$, what must we call the former? Answers: $4x$ and $45x$.

Example. If A, working alone, can do a piece of work in 3 days, and B, working alone, can do it in 4 days, in what time will they do it working together?

Let w represent the work. Then, in one day,

$$A \text{ does } \tfrac{1}{3}\, w = \tfrac{4}{12}\, w,$$
$$B \text{ does } \tfrac{1}{4}\, w = \tfrac{3}{12}\, w.$$

Consequently, working together, they will do $\frac{7}{12}\, w$ in one day. But the quantity of work done is proportional to the time employed upon it, when all other circumstances are the same.

$$\text{Therefore } \tfrac{7}{12}\, w : w :: 1 \text{ day} : \text{time required.}$$

The time required is therefore the blank term, which, found by the rule, is $\frac{12}{7}$ of a day, or $1\frac{5}{7}$.

Exercise 3. If a gardener alone could dig a piece of ground in 10 days, and his apprentice alone could dig it in 12 days, in what time will both finish it, working together? Ans. In $5\frac{5}{11}$ days.

4. There are three pipes which communicate with the same cistern. The first, when running alone, takes 24 hours to fill the cistern; the second, 20 hours; and the third, 18 hours. In what time will they fill it all running together? Ans. In $6\frac{42}{53}$ hours.

5. A man and his wife found that a cheese, of the ordinary size used by them, usually lasted a fortnight when they were both at home; but when the husband was absent, it lasted his wife 5 weeks and a day. How long would it serve the man when his wife was absent? Ans. $22\frac{10}{11}$ days.

Problem II.

To find a mean Proportional between two given Quantities.

Rule. Multiply the two given quantities together and extract the square root of their product.

Example. Find a mean proportional between $4a^2$ and $9b^2$.

$$\sqrt{(4a^2 \times 9b^2)} = \sqrt{36a^2b^2} = 6ab.$$

Exercise 1. Find a mean proportional between 49 and 144. Ans. 84.

2. Between $9xy^3$ and $64x^3y$. Ans. $24x^2y^2$.

3. Between $(a+x)^2$ and $(a-x)^2$. Ans. a^2-x^2.

CHAPTER XIII.

EQUATIONS.

DEFINITIONS.

1. When two quantities are equal, the expression of that equality is called an *Equation*.

2. When two quantities are *necessarily* equal (that is,

when they are equal under every value that can be assigned to each letter, or when they consist of equal combinations of particular numbers), the quantities are said to be *Identical*, and sometimes the equation itself, expressing their equality, is designated by the same word. Thus $6+4 = 2 \times 5$, and $(a+x) \times (a-x) = a^2 - x^2$, are identical equations, or equations of identical quantities.

3. When an equation (not identical or impossible) contains an unknown quantity mixed with known quantities (as $3x^2 - 5x = 2$), its value is said to be *implicitly* expressed by the equation: that is, the equation contains enough to render the value of the unknown quantity determinate, although it does not directly inform us what that value is. But when a quantity previously unknown is brought by any means to stand alone on the one side, and known quantities only on the other (as $x = a+b$), we then have the value of the previously unknown quantity *explicitly* declared.

4. Equations are applied to various uses; but their peculiar purpose, when treated as a separate branch of Algebra, is to enable us to obtain explicitly the values of unknown quantities, from given equations in which those values are only implicitly expressed. Thus, if we have given the equation, $\frac{1}{2}x + \frac{1}{3}x = 10$, that equation, after certain operations, is transformed into $x = 12$. We are then said to have *solved* or *resolved* the given equation; and the process by which we do so, is called the *Solution* or *Resolution of Equations.*

5. When equations are thus used for the determination of unknown quantities, they are distinguished by different names, according to the highest power of any unknown quantity involved in them. If an equation, when cleared of fractions and roots, contains no power of any unknown quantity higher than the first, it is called a *Simple Equation;* if none higher than the second, a *Quadratic Equation;* if it contains the third power and no higher, it is called a *Cubic Equation;* and if the fourth power and no higher, a *Biquadratic Equation.*

For the solution of equations generally we must have recourse to one or more of the processes described in the following rules. Some of them are first principles, or nearly so: the others are derived from these, or, in fact, are mere repetitions of them in different forms, one form being sometimes more convenient, sometimes another.

GENERAL RULES.

1. Equal quantities* may be added to both sides.
2. Equal quantities* may be taken from both sides.
3. Both sides may be multiplied by equal quantities.*
4. Both sides may be divided by equal quantities.*
5. Both sides may be raised to the same power.
6. The same root of both sides may be extracted.
7. For any quantity its equal may be substituted.
8. Any term may be transposed from the one side to the other, if its sign be changed.
9. *All* the terms on *both* sides may be transposed without changing any of the signs.
10. *All* the signs of *both* sides may be changed, without transposing any of the terms.

Any of these operations may be performed, and the equality will still be preserved.

The OBJECTS to be attained by the application of these rules, are—

1*st*. To bring all the terms containing unknown quantities to one side (that side being almost invariably the left), and all the other terms to the other side.

2*d*. To produce an equation containing only one unknown quantity.

3*d*. To clear the unknown quantity of all co-efficients, divisors, and exponents of powers or roots, so that, at last, it may stand out alone on the one side, and its value in known quantities on the other.

These objects are not always to be attained in the order here described, but sometimes in one order, at other times in another, according to circumstances.

In the case of simple equations with only one unknown quantity, the circumstances and modes, in which the different rules should be employed, will be almost obvious if we keep in mind the *objects* to be attained; but they will be still better understood from the examples. In the cases of equations involving more than one unknown quantity, and of quadratic and higher equations, *particular rules* become necessary.

When there is only one unknown quantity, one equation is always sufficient† to determine its value: but if there are more unknown quantities than one, there must be as many independent equations.

---

* Or the same quantity.

† Unless it be an identical equation.

When the value of an unknown quantity is found, it is often useful to *prove* the result, by inserting that value, instead of the unknown quantity, in the original equation. If the value found is correct, the equation becomes identical.

CHAPTER XIV.

SIMPLE EQUATIONS.

PROBLEM I.

To resolve a simple Equation containing one unknown Quantity.

RULE. Apply one or more of the *General Rules* given in the last chapter, in such a way as to attain the *Objects* there specified. When at last the unknown quantity has been brought to stand alone on the left side, and its value, in known quantities, on the right, the equation is resolved.

EXAMPLE 1. Having given $x - 5 = 7$, to find the value of x.

Adding 5 to each side, by Rule 1, we have

$$x = 12.$$

PROOF. $12 - 5 = 7$.

NOTE 1. In subsequent examples it will be unnecessary to give the proof; but it may be useful for the learner to apply it in many, or all, of the exercises.

EXERCISE 1. Given $x - 17 = 24$, to determine x. Ans. $x = 41$.

2. Given $y - 36 = -19$, to find y. Ans. $y = 17$.

NOTE 2. In the following examples and exercises, when we say merely that a certain equation is given, it is understood that we are to determine the value or values of the unknown quantity or quantities expressed by the letters x, y, z.

EXAMPLE 2. Given $x + b = a$.

Subtracting b from each side by Rule 2, or transposing it from the left to the right by Rule 8, we have

$$x = a - b.$$

EXERCISE 3. Given, $x + 15 = 20$. Ans. $x = 5$.

4. Given, $x + 3c = 5b$. Ans. $x = 5b - 3c$.

Example 3. Given, $5a - z = 2b$.

By Rule 8, $\quad -z = -5a + 2b$.

Then, by R. 10, $\quad z = 5a - 2b$.

Exercise 5. Given, $128 - x = 46$. Ans. $x = 82$.

6. Given, $36 - y = -15$. Ans. $y = 51$.

Example 4. Given, $ax = c$.

Dividing both sides by a, by Rule 4,

$$x = \frac{c}{a}.$$

Exercise 7. Given, $7x = 35$. Ans. $x = 5$.

8. Given, $52x = 130$. Ans. $x = 2\frac{1}{2}$.

Example 5. Given, $\frac{x}{a} = b - c$.

Multiply both sides by a, by Rule 3, the fraction being multiplied according to the rule in Note 3 of Pr. IX, Ch. IX. Then we have

$$x = ab - ac.$$

Exercise 9. Given, $\frac{x}{5} = 8$. Ans. $x = 40$.

10. Given, $\frac{y}{b} = a + b$. Ans. $y = ab + b^2$.

Example 6. Given, $125 - \frac{3x}{4} = 41$.

Proceeding by successive operations, we have—

By Rule 2, $\quad 84 - \frac{3x}{4} = 0$.

By Rule 3, $\quad 336 - 3x = 0$.

By Rule 8, $\quad -3x = -336$.

By Rule 10, $\quad 3x = 336$.

By Rule 4, $\quad x = 112$.

Exercise 11. Given, $4x + 56 = 17x - 35$. Ans. $x = 7$.

12. Given, $\frac{5x}{12} = 45$. Ans. $x = 108$.

13. Given, $2y - 3a = 10b$. Ans. $y = \frac{3}{2}a + 5b$.

14. Given, $\frac{7z}{10} - 63 = -42$. Ans. $z = 30$.

15. Given, $3a - \frac{3x}{20} = 8c$. Ans. $x = 20a - 53\frac{1}{3}c$.

16. Given, $\frac{6z - 4}{3} - 2 = \frac{18 - 4z}{3} + z$. Ans. $z = 4$.

Note 3. When several fractions occur with different denominators, and all involving the unknown quantity, the fractions may be changed into whole numbers either by multiplying by their denominators successively, or by multiplying at once by any common multiple of all the denominators. The shortest mode, and perhaps the easiest, is to multiply at once by the *least* common multiple of the whole, observing the rule contained in Note 4 to Problem IX of Chapter IX.

Example 7. Given, $\frac{x}{2}+\frac{x}{3}-\frac{x}{5}=38$.

Here the least common multiple of all the denominators is their product 30. We therefore multiply at once by 30, by Rule 3, and find

$$15x+10x-\ \ 6x=1140.$$
$$\text{That is}\qquad 19x=1140.$$
$$\text{By Rule 4,}\qquad x=60.$$

Exercise 17. Given, $\frac{x}{2}+\frac{2x}{3}=147$. Ans. $x=126$.

Example 8. Given, $\frac{3z}{4}-\frac{z}{6}+\frac{5z}{9}=100$.

The least common multiple of 4, 6, and 9 being 36, we multiply all the terms by that, and we have

$$27z-6z+20z=3600,$$
$$\text{Or}\qquad 41z=3600.$$
$$\text{By Rule 4,}\qquad z=87\tfrac{33}{41}.$$

Exercise 18. Given, $\frac{5x}{6}-\frac{7x}{10}-64=0$. Ans. $x=480$.

19. Given, $\frac{x}{6}-\frac{x}{4}-9=\frac{x}{3}-\frac{x}{2}$. Ans. $x=108$.

20. Given, $\frac{20x}{21}-75=\frac{5x}{12}+\frac{5x}{16}$. Ans. $x=336$.

21. Given, $\frac{y+2}{5}=\frac{2y-6}{3}-6$. Ans. $y=18$.

22. Given, $\frac{3x-33}{4}-12=\frac{21-x}{2}-\frac{8x+45}{6}$.

Ans. $x=9$.

23. Given, $\frac{30}{y}=\frac{40}{y}-\frac{5}{6}$. Ans. $y=12$.

24. Given, $8-\frac{35}{z}=\frac{28}{2z}+\frac{21}{3z}$. Ans. $z=7$.

25. Given, $\frac{y}{4-y}-7=24$. Ans. $y=3\tfrac{7}{8}$.

***Note** 4.* **When a fraction, having its numerator a com-**

pound quantity, is preceded by a negative sign, we must either transpose the whole fraction, changing its sign, or we must keep in mind, that, in reducing it to an integral form, the vinculum is removed (See Chap. I, 33), and must therefore change all the signs of the numerator (See Chap. IV, Rule 4.)

EXAMPLE 9. Given, $6x - \frac{2x-3}{5} = 51$.

Transposing the fraction, $6x = 51 + \frac{2x-3}{5}$.

Multiplying by 5, $30x = 255 + 2x - 3$.
Hence $28x = 252$, and $x = 9$.

Or thus:—Multiplying by 5 without transposing, and observing Rule 4 of Ch. IV,

$$30x - 2x + 3 = 255.$$

Hence $28x = 252$, and $x = 9$.

EXERCISE 26. Given, $\frac{5x}{2} - \frac{3x+5}{4} = 11$. Ans. $x = 7$.

27. Given, $5x + \frac{7x+2}{3} - \frac{5x-6}{7} = 28$. Ans. $x = 4$.

28. Given, $\frac{3x-6}{4} - \frac{3x-8}{3} = 10\frac{2}{3} - \frac{54+4x}{9}$.
Ans. $x = 18$.

EXAMPLE 10. Given, $\sqrt[3]{3x} = 12$.
By Rule 5, $3x = 1728$.
By Rule 4, $x = 576$.

EXERCISE 29. Given, $\sqrt{7x} = 35$. Ans. $x = 175$.
30. Given, $5\sqrt{x} + 12 = 27$. Ans. $x = 9$.
31. Given, $\sqrt{(y-35)} = 15$. Ans. $y = 260$.
32. Given, $\sqrt[3]{(4z+3)} = 3$. Ans. $z = 6$.
33. Given, $\sqrt{(19x+13)} = \sqrt{(59-4x)}$. Ans. $x = 2$.
34. Given, $\sqrt{(y+9)} = 1 + \sqrt{y}$. Ans. $y = 16$.
35. Given, $\frac{10z-18}{\sqrt{5z+3}} - 2 = \sqrt{5z-3}$. Ans. $z = 5$.

EXAMPLE 11. Given, $x+3 : x-3 :: 5 : 4$.
By Ch. XII, Th. I, $5x - 15 = 4x + 12$.
Hence, $x = 27$.

EXERCISE 36. Given, $2x : x+20 :: 9 : 10$.
Ans. $x = 16\frac{4}{11}$.

37. Given, $\frac{10+x}{5} : \frac{4x-9}{7} :: 14 : 5$. Ans. $x = 4$.

38. Given, $\frac{17-4x}{4} : \frac{15+2x}{3} - 2x :: 5 : 4$.

Ans. $x = 3$.

PROBLEM II.

To find the Values of two unknown Quantities, from two independent Equations.

RULE. Having simplified and arranged each equation, if necessary, by changing fractional quantities into integers, and collecting all the terms containing one of the unknown quantities into one term, all those containing the other unknown quantity into another term, and all the known quantities into a single group, as in the following examples, and by processes exactly the same as those in the preceding problem, each equation will then appear in the form, $ax + by = c$, in which a, b, and c may be any known quantities whatever, simple or compound. We then, from the two equations combined, produce one resulting equation containing only one unknown quantity, from which the value of that quantity is found by Problem I. This resulting equation may be obtained by any one of the three methods presently to be described.

When the value of the one unknown quantity has been found, it is inserted, instead of that quantity, in either of the two original simplified equations, and we obtain another equation, containing only the other unknown quantity, also to be resolved by Problem I.

METHOD FIRST. In either equation treat one of the unknown quantities as if it were known, and find the value of the other in terms of the first and of known quantities. Having found that value, insert it, instead of its equivalent letter, in the other equation. That equation will then contain only one unknown quantity.

METHOD SECOND. From the two equations, treated separately, find two separate values for one of the two unknown quantities in terms of the other and of known quantities, in precisely the same manner as we found one value from one equation by Method First. The two values thus found being, of necessity, equal, give us a resulting equation containing only one unknown quantity.

METHOD THIRD. Multiply each side of both equations by such numbers as will equalize the two co-efficients of *one of the* unknown quantities. We have thus two new equations, the sum or difference of which is our resulting equation containing only one unknown quantity.

NOTE 1. The quantity which disappears in the resulting equation, is said to be *eliminated*.

NOTE 2. In following the third method, it may be that the two co-efficients of one of the unknown quantities in the given equations are equal at first (as in Example 2): in that case no multiplication is necessary. Or one of the given co-efficients may be a multiple of the other (as in Example 3); in that case only one equation requires to be multiplied. *In general*, the two co-efficients will be rendered equal if we use each as a multiplier of the other equation (as in Example 1): but the operation is facilitated if we previously divide the co-efficients by their greatest common measure (if they have a common measure greater than 1), and use the quotients as the multipliers (as in Example 4): in this case the two co-efficients obtained by the multiplication are each the least common multiple of the two original co-efficients. Similar modifications may be made in Methods First and Second.

EXAMPLE 1.

Given $\left\{\begin{array}{l} 2x-3y=7, \\ 3x-4y=12, \end{array}\right\}$ to find x and y.

By Method First.

From Equation 1*, $x=\frac{7+3y}{2}$.

Inserting this value for x in Equation 2,

$$3\times\frac{7+3y}{2}-4y=12.$$

That is, $\frac{21+9y}{2}-4y=12.$

Or $21+9y-8y=24.$

$$\therefore y=3.$$

By Method Second.

From Eq. 1, $x=\frac{7+3y}{2}$.

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* That is, the first of the two given equations.

$$\text{From Eq. 2, } x = \frac{12 + 4y}{3}.$$

$$\therefore \frac{7 + 3y}{2} = \frac{12 + 4y}{3}.$$

$$\text{Multiplying by 6, } 21 + 9y = 24 + 8y.$$

$$\text{Hence } y = 3.$$

By Method Third.

Multiply Eq. 1 by 3. Then $6x - 9y = 21$.
Multiply Eq. 2 by 2 $6x - 8y = 24$.

Subtract the first of the two new equations from the second. Then

$$y = 3.$$

Having now found $y = 3$ by each of the three methods, the remainder of the process is the same for all three: that is, we insert 3 for y in either of the two given equations—say in the first, and we have the resulting equation,

$$2x - 9 = 7.$$

$$\text{Hence } x = 8.$$

Example 2. Given $\begin{cases} 4x + 3y = \ \ 51, \\ 4x - 5y = -21, \end{cases}$ to determine x and y *by Method Third.*

$$\text{Subtracting Eq. 2 from Eq. 1, } 8y = 72.$$

$$\text{Hence we find } y = 9, \text{ and } x = 6.$$

Example 3. Given $\begin{cases} 5x - \frac{3}{2}y = 39, \\ 10x - \frac{4}{7}y = 112, \end{cases}$ to determine x and y *by Method Third.*

$$\text{Doubling Eq. 1, } 10x - 3y = 78.$$

$$\text{Subtract this from Eq. 2. } 3y - \tfrac{4}{7}y = 34.$$

$$\text{Hence } y = 14, \text{ and } x = 12.$$

Example 4. Given $\begin{cases} 12x + \ \ 3y = 114, \\ \ \ 8x + 10y = 156, \end{cases}$ to find x and y *by Method Third.*

The greatest common measure of 12 and 8 is 4. $12 \div 4 = 3$; and $8 \div 4 = 2$. Therefore we multiply the two given equations respectively by 2 and 3, and obtain

$$24x + 6y = 228.$$

$$24x + 30y = 468.$$

$$\text{By subtracting, } 24y = 240.$$

$$\text{Hence, } y = 10, \text{ and } x = 7.$$

Note 3. The third method of operation is usually the most pleasant, from the absence of fractions. It is also *frequently superior* to the others in point of facility, and seldom inferior. It is therefore the best mode, although the others may be practised.

Example 5. Given $\left\{\begin{matrix} 13x-4y=14, \\ 15x+2y=36, \end{matrix}\right\}$ to find the values of x and y *by Method Third.*

Here we find it the easiest mode to eliminate y. We therefore multiply the second equation by 2, and *add* the product to the first. We then have

$$43x=86.$$

$$\text{Hence } x=2, \text{ and } y=3.$$

Note 4. In peculiar circumstances we may occasionally deviate a little from the direct process by either of the three methods—sometimes with advantage. Thus, returning to Example 1, if we had begun by subtracting Eq. 1 from Eq. 2, we should have had $x-y=5$, and $x=y+5$, which would have been a simpler value to substitute for x by Method First, or to use as one of the two values of x by Method Second. We might also, if we had chosen, have used it instead of Eq. 2, in following the Third Method.

Exercise 1. Given $\begin{cases} x+y=25, \\ x-y=\ \ 7. \end{cases}$ Ans. $\begin{cases} x=16, \\ y=\ \ 9. \end{cases}$

2. Given $\begin{cases} 6x+4y=72, \\ 6x-3y=30. \end{cases}$ Ans. $\begin{cases} x=8, \\ y=6. \end{cases}$

3. Given $\begin{cases} 3x+7y=117, \\ 12x-5y=\ \ 72. \end{cases}$ Ans. $\begin{cases} x=11, \\ y=12. \end{cases}$

4. Given $\begin{cases} 6x-7y=-9, \\ 7x-6y=-4. \end{cases}$ Ans. $\begin{cases} x=2, \\ y=3. \end{cases}$

5. Given $\begin{cases} 10x+13y=36, \\ 15x+19y=53. \end{cases}$ Ans. $\begin{cases} x=1. \\ y=2. \end{cases}$

6. Given $\begin{cases} 13x-2y=58, \\ 15x-2y=70. \end{cases}$ Ans. $\begin{cases} x=\ \ 6, \\ y=40. \end{cases}$

7. Given $\begin{cases} 7x+3y=122, \\ 9x+\ \ y=144. \end{cases}$ Ans. $\begin{cases} x=15\frac{1}{2}, \\ y=\ \ 4\frac{1}{2}. \end{cases}$

8. Given $\begin{cases} 73x+4y=446, \\ 50x-6y=288. \end{cases}$ Ans. $\begin{cases} x=6. \\ y=2. \end{cases}$

9. Given $\begin{cases} x+y=20, \\ x\div y=\ \ \frac{2}{3}. \end{cases}$ Ans. $\begin{cases} x=\ \ 8, \\ y=12. \end{cases}$

10. Given $\begin{cases} x^2-y^2=21, \\ x\ -y\ =\ \ 1. \end{cases}$ Ans. $\begin{cases} x=11. \\ y=10. \end{cases}$

11. Given $\begin{cases} 7\frac{1}{2}y+13\frac{1}{3}z=660, \\ 8\frac{1}{3}y+\ \ 5\frac{1}{2}z=398. \end{cases}$ Ans. $\begin{cases} y=24, \\ z=36. \end{cases}$

12. Given $\begin{cases} \dfrac{2x+3y}{6}+\dfrac{x}{3}=8, \\ \dfrac{7y-3x}{2}-y=11. \end{cases}$ Ans. $\begin{cases} x=6, \\ y=8. \end{cases}$

13. Given $\begin{cases} \dfrac{7+x}{5}-\dfrac{2x-z}{4}=3z-5, \\ \dfrac{5z-7}{2}+\dfrac{4x-3}{6}=18-5x. \end{cases}$ Ans. $\begin{cases} x=3, \\ z=2. \end{cases}$

14. Given $\begin{cases} 4y-3x=12, \\ 3x : 4y :: 5 : 7. \end{cases}$ Ans. $\begin{cases} x=10, \\ y=10\frac{1}{2}. \end{cases}$

15. Given $\begin{cases} x+y : x-y :: 3 : 1, \\ 3x+4 : 6y-4 :: 5 : 3. \end{cases}$ Ans. $\begin{cases} x=5\frac{1}{3}, \\ y=2\frac{2}{3}. \end{cases}$

16. Given $\begin{cases} \dfrac{5x+13}{2}-\dfrac{8y-3x-3}{6}=9+\dfrac{7x-3y}{3}, \\ \dfrac{x+7}{6} : 2x+\dfrac{3y-8}{8} :: 4 : 21. \end{cases}$

Ans. $x=5$, $y=4$.

PROBLEM III.

To find the Values of three or more unknown Quantities from as many independent Equations.

RULE. There are three methods of proceeding, corresponding to the three methods of the last Problem, by any one of which a new set of equations is produced, one fewer in number, and entirely cleared from one of the unknown quantities. By repeating the same process with the new set of equations, another of the unknown quantities is eliminated; and so on till only one remains, the value of which we find by Problem I.

EXAMPLE. Given $\left.\begin{cases} x+y+z=+6, \\ 2x+3y-4z=-4, \\ 3x+8y-6z=+1, \end{cases}\right\}$ to find x, y, and z.

By Method First.

From Eq. 1, $x=6-y-z$.

Inserting this value in Eqs. 2 and 3,

we have $2(6-y-z)+3y-4z=-4$,

and $3(6-y-z)+8y-6z=+1$.

These two equations, simplified, become

$$y-6z=-16, \text{ and}$$
$$5y-9z=-17.$$

We have thus found two equations containing only y and z.

By Method Second.

From the three given equations, separately, we obtain

$$x=6-y-z,$$
$$x=\frac{-4-3y+4z}{2},$$
$$x=\frac{1-8y+6z}{3}.$$

And, equating these with each other, two by two,

$$\text{we have } 6-y-z=\frac{-4-3y+4z}{2},$$
$$\text{and} \quad 6-y-z=\frac{1-8y+6z}{3}.$$

These two equations, simplified, and cleared of fractions, become

$$y-6z=-16, \text{ and}$$
$$5y-9z=-17, \text{ as before.}$$

By Method Third.

$$\text{From Eq. 1, } 2x+2y+2z=12,$$
$$\text{and} \quad 3x+3y+3z=18.$$

Subtract these, severally, from Eqs. 2 and 3.

$$y-6z=-16, \text{ and}$$
$$5y-9z=-17, \text{ as before.}$$

Having thus, by each of the three different methods, obtained the same two resulting equations containing only y and z, we proceed to find the values of y and z by Problem II,—viz., 2 and 3,—and then, inserting these values for y and z in Eq. 1, we obtain another resulting equation containing only x, viz. $x+5=6$, from which we find $x=1$.

Note. In working this example we might have deviated from the direct process with advantage, by first subtracting Eq. 1 from Eq. 2, and then Eq. 2 from Eq. 3. The two new equations, along with Eq. 1, would have been more easily treated by any of the three methods, than our three original equations.

EXERCISES.

1. Given $\begin{cases} x+y+z=22, \\ x+y-z=20, \\ x-y+z=\ 4. \end{cases}$ Ans. $\begin{cases} x=12, \\ y=\ 9, \\ z=\ 1. \end{cases}$

2. Given $\begin{cases} 2x+2y-3z=18, \\ 3x-2y+\ z=20, \\ 4x-4y-\ z=\ 2. \end{cases}$ Ans. $\begin{cases} x=10, \\ y=\ 8, \\ z=\ 6. \end{cases}$

3. Given $\begin{cases} x-\ y+\ z=\ 8, \\ x+2y+3z=\ 84, \\ \frac{1}{2}x+\frac{1}{3}y-\frac{1}{4}z=-1. \end{cases}$ Ans. $\begin{cases} x=\ 0, \\ y=12, \\ z=20. \end{cases}$

4. Given $\begin{cases} x-y=2, \\ y+z=9, \\ x-z=4. \end{cases}$ Ans. $\begin{cases} x=7\frac{1}{2}, \\ y=5\frac{1}{2}, \\ z=3\frac{1}{2}. \end{cases}$

5. Given $\begin{cases} \frac{1}{3}(x-y)+z=22, \\ x+\frac{1}{2}(y+z)=16, \\ (x-y+z)\div z=4. \end{cases}$ Ans. $\begin{cases} x=+18, \\ y=-15, \\ z=+11. \end{cases}$

CHAPTER XV.

QUESTIONS PRODUCING SIMPLE EQUATIONS.

The method of proceeding, in finding answers to questions by Algebra, is, to represent the unknown quantities by one or more of the letters x, y, z, and then to express, in one or more equations, the different facts or suppositions stated regarding them. When these equations contain no powers higher than the first, the remainder of the operation will be the same as one or other of the processes described in the last chapter.

Some skill is occasionally required in reducing the data of the questions to an Algebraic form, since that may often be done in various ways.

Sometimes we represent the unknown quantity—not by a single letter, but by a letter with a co-efficient, as in Example 2, or with some other qualification.

At other times it is best to regard as the unknown quantity—not that which is directly sought, but some other from which it may be easily deduced. An instance *of this will be* seen in Example 3, in which x is made to represent the time instead of the distance sought. The object of doing so is to avoid fractions.

Sometimes we are obliged to use several letters, to represent the different quantities or numbers sought, as in Example 6; but in many instances in which this appears, at first sight, to be necessary, we find, on further examination, that it may be avoided, and that we obtain a more elegant solution by using only one letter, as in Examples 4 and 5.

Most of the questions, producing simple equations, may be resolved by pure Arithmetic, but that is no hindrance to their forming good examples in Algebra. The latter, indeed, is almost always the easier method, the arithmetical process requiring more consideration and more ingenuity.

Example 1. What number is that, the double of which as much exceeds 25 as its half falls short of it?

Let x be the number.

$$\begin{aligned} \text{Then, } 2x-25 &= 25-\frac{x}{2}. \\ \text{Hence} \qquad 5x &= 100, \\ \text{and} \qquad x &= 20. \end{aligned}$$

Example 2. What number is that, whose half, added to its third part, is 10?

Let $6x$ be the number.

$$\begin{aligned} \text{Then, } 3x+2x &= 10. \\ \text{Hence} \qquad x &= 2, \\ \text{and} \qquad 6x &= 12. \end{aligned}$$

Example 3. A traveller sets out from Milton to Newton, 35 miles distant, walking at the rate of 4 miles per hour. Another, at the same time, sets out from Newton to Milton at the rate of 3 miles per hour. In what part of the road do they meet?

Let the number of hours they walk before meeting each other be represented by the letter x: then the first will have walked $4x$ miles, and the second, $3x$. But these two distances make up the whole distance between the two places.

$$\begin{aligned} \because \quad 4x+3x &= 35. \\ \text{Hence} \qquad x &= 5, \\ 4x &= 20, \\ \text{and} \qquad 3x &= 15. \end{aligned}$$

That is, they meet 20 miles from Milton and 15 from Newton.

Example 4. What two parts, in the proportion of 4 to 5, make up a line of 100 feet?

Let the two parts be $4x$ and $5x$.

$$\text{Then, } 4x + 5x = 100.$$
$$\text{Hence } x = 11\tfrac{1}{9},$$
$$4x = 44\tfrac{4}{9},$$
$$5x = 55\tfrac{5}{9}.$$

Or thus.—Let x be the smaller part: then $100 - x$ will be the greater; and

$$x : 100 - x :: 4 : 5.$$
$$\therefore\ 5x = 400 - 4x.$$
$$\text{Hence, } x = 44\tfrac{4}{9}, \text{ and}$$
$$100 - x = 55\tfrac{5}{9}.$$

Example 5. To divide the number 90 into four such parts that, if the first be increased by 5, the second decreased by 4, the third multiplied by 3, and the fourth divided by 2, the results, in all the cases, shall be exactly the same.

Let x represent the result in each case. Then $x - 5$ is the first part; $x + 4$, the second; $\frac{x}{3}$ the third; and $2x$ the fourth.

$$\text{But } (x - 5) + (x + 4) + \frac{x}{3} + 2x = 90.$$

Hence $13x = 273$, and $x = 21$; and the four parts are 16, 25, 7, and 42.

Example 6. What is that fraction which would just be equal to $\frac{1}{2}$ if the numerator were increased by 3; but only to $\frac{1}{4}$ if the numerator were diminished by 2?

Let x and y be the numerator and denominator, and consequently, $\frac{x}{y}$ the fraction.

$$\text{Then } \frac{x + 3}{y} = \frac{1}{2}, \text{ and } \frac{x - 2}{y} = \frac{1}{4}.$$

From these two equations we find $x = 7$, $y = 20$, and $\frac{x}{y} = \frac{7}{20}$.

Exercise 1. What is that number which, with 17 added to it, will make 55? Ans. 38.

2. What number is that, whose half exceeds its third part by 17? Ans. 102.

3. A boy being asked how many marbles he had, said *he would answer* no such question, but that it was a mat-

ter of indifference to him whether he gained 15 more, or doubled what he had and then lost 10. Can you compute, from that hint, how many he actually had? Ans. 25.

4. An election, at which there were only two candidates, was gained by Mr Smith, who had a majority of 75 over Mr Jones: the number of voters in the interest of Jones was just two-thirds of the number who supported Smith: What was the whole number of voters?
Ans. 375.

5. The sum of two numbers is 154, and their difference 42: What are they? Ans. 98 and 56.

6. A charitable gentleman distributed the contents of his purse, which contained 23 shillings, among a poor family, consisting of a man, his wife, and two children. He gave the man twice as much as his wife, and to the oldest child one-third of what he gave to his father. There was then a shilling over, which he gave to the youngest child: How much did each receive?
Ans. 12s., 6s., 4s., and 1s., respectively.

7. There are three villages, A, B, and C, situated triangularly with regard to each other. A letter-carrier goes round them once a-day, and finds that, when he walks at his usual pace of 3 miles an hour, the last stage from C to A takes him 20 minutes less time than the first, from A to B, but 50 minutes more than the intermediate stage, from B to C, and that, after deducting the time lost in stoppages, it takes him just 6 hours to go all round: What were the distances of the villages from one another?
Ans. $7\frac{1}{2}$, 4, and $6\frac{1}{2}$ miles.

8. A gentleman, after travelling 12 hours without stopping, found that, if he had travelled 3 miles an hour faster, he would have accomplished the journey in two hours less of time: At what rate did he travel?
Ans. 15 miles per hour.

9. Two boys, who made their living by selling nuts, commenced the week with the same sum; but, when they met on the Saturday night, the one found that, after paying for his maintenance, he had gained a half-crown; and the other, that he had lost 1s. 6d. The consequence was, that the former had now three times as much as the latter. What did they begin the week with? Ans. 3s. 6d. each.

10. If you divide a certain number by 9, and add together divisor, dividend, and quotient, their sum shall be 59. The number is required. Ans. 45.

11. Divide 1000 into two such parts, that 9 times the larger shall exceed 13 times the smaller by 970.
Ans. 635 and 365.

12. Divide the number 100 into two such parts, that half the one added to one-seventh of the other shall just make 20. Ans. 16 and 84.

13. Find three numbers, such that their sums, taken two by two, shall be 11, 12, and 13. Ans. 5, 6, and 7.

14. A certain reading-room is maintained at the joint expense of the subscribers. At the end of the first year they find they have each £1, 4s. to pay; but one of them remarked correctly that, if there had been seven more of them they would have got off at the rate of a sovereign a piece. Compute the number of members. Ans. 35.

15. A man departs on a journey, walking at the uniform rate of 3 miles an hour; and, two hours later, another sets out after him, riding, at 7 miles an hour. At what distance on the road will the latter overtake the former? Ans. $10\frac{1}{2}$ miles.

16. How long could the rider, described in the preceding question, have delayed his departure, after the walker set out, so as to overtake him at the 14th milestone?
Ans. $2\frac{2}{3}$ hours.

17. Two numbers are to each other in the proportion of 4 to 5; but if 5 be taken from the smaller, and 11 added to the greater, the one is then double of the other. What are the numbers? Ans. 28 and 35.

18. A man and his wife were married at the respective ages of 40 and 20 years. How old will the man be when his wife's age becomes three-fourths of his own?
Ans. 80 years.

19. Find two numbers such, that, if 6 be added to each, they shall be to each other as 4 to 5; and, if 4 be taken from each, they shall be to each other as 2 to 3.
Ans. 14 and 19.

20. A lady, on the supposed anniversary of the birth of our Saviour, presented all the wives and children of her cottagers with donations of money. She gave the women a sovereign each, and the children a half-crown. Having done so, she found that she had bestowed 200 donations, and expended £58, 5s. What were the respective numbers of women and children? Ans. 38 and 162.

21. A farmer sold, in the market, a number of sheep and cattle, there being seven sheep for every two cattle. The latter were sold, on an average, at 8 guineas a-piece, and the former at 25 shillings. After paying 15s. 6d. of expenses, he brought away £305 : 16 : 6. How many had he of each kind? Ans. 24 cattle and 84 sheep.

22. A lady's ring cost 7 guineas; her brooch cost as much as her ring and half her bracelets, and the bracelets cost as much as the ring and brooch together. What was the cost of the whole? Ans. 56 guineas.

23. A tea-dealer considers that 2 ℔ of his best green tea is worth three of his best black. He mixes them together in equal proportions, and sells the mixture at 6s. 3d. per pound. At what rate would he sell the black and green separately? Ans. At 5s. and 7s. 6d.

24. What are the two numbers, whose sum is 133, and quotient 18? Ans. 126 and 7.

25. Find two numbers whose difference shall be 1, and the difference of their squares 19. Ans. 9 and 10.

26. Divide 100 into two such parts that the difference of their squares may be 200. Ans. 51 and 49.

27. There are three numbers such, that the first, with half the sum of the other two, makes 46; the second, with one-third of the other two, makes 30; and the third, with one-fourth of the first and second, amounts to 29. Calculate the numbers. Ans. 30, 14, and 18.

28. A school was divided into three classes, the number in the first class being to that in the second as 3 to 4, and the latter to the number in the third, as 7 to 8. There were 81 in all: how many were in each class?
Ans. 21, 28, and 32.

29. Two ships' boats were sent out on a whaling excursion, with crews in the proportion of 3 to 5. Meeting sometime after, they considered it would be better to divide the hands equally between the two boats, and did so by removing 6 men from the one to the other. What was the whole number sent out? Ans. 48.

30. Two brothers, John and George, go to school at the beginning of the year with 5 guineas a-piece of pocket-money. At the end of the year John finds that he has spent twice as much as George and 5 shillings more, and that he has remaining 5 shillings less than half of what George still possesses. How much does each return home with? Ans. £1, 10s. and £3, 10s.

31. A gentleman left £8400 to be divided among his four nephews in certain portions named, and directed his house to be sold by auction. As the four nephews were returning together from the sale, Edward observed, that if his uncle had left him twice as much as he did, he could have bought the house with the sum. "He left me enough," said Richard, "to buy it twice over if I were disposed." "It would have taken Harry's and mine put together to purchase it," replied Alfred. "I could have paid for it with my own share," exclaimed Harry, "if Dick had just been generous enough to add to it the third part of his." What was the house sold for?

Ans. £2400.

32. A son, asking his father how old he was, received the following reply:—"Seven years ago I was four times as old as you; but seven years hence, if you and I live, my age will then be only double yours." It is required, from this information, to satisfy the son's curiosity.

Ans. The father's age was 35.

33. An itinerant orange-vender bought a quantity of oranges for sale, at the rate of 5 for twopence. He then arranged the good and the bad in two separate baskets, containing equal numbers, and sold the one basketful at 3 a penny and the other at 2 a penny. In selling them he met another orange-vender, who laughed at his simplicity, and said he would have no profit upon them; but, when he had sold the whole, he found he had gained sixpence. Please to calculate, from these data, how many oranges he bought and sold. Ans. 30 dozen.

34. Two men and a boy concerted to steal a parcel of money, and, having succeeded, met to divide the contents. It was found to contain £4 : 2 : 6 in silver. One of the men, who divided it, took to himself twice as much as he gave to the boy; and, to keep the other man quiet, gave him a half-crown more than would have been his share if the money had been equally divided among the three. How much did the one man get more than the other?

Ans. 5 shillings.

35. Two women occupy adjacent stalls at a fair. The one makes four sales for the other's three, and has sold three dozen articles before the other commences; but then, she only draws as much at three sales as her neighbour does at two. How many articles must the latter dispose of before she have as much money in hand as the former?

Ans. 216.

CHAPTER XVI.

QUADRATIC EQUATIONS.

DEFINITIONS.

A *Quadratic Equation*, as has been already defined, is one which contains the second power of the unknown quantity, and no higher. When the second power occurs alone without the first, the equation is called a *Pure Quadratic*, as $ax^2 = b^3$. When both the first power and the second enter into it, it is called an *Adfected Quadratic*, as $x^2 + bx = c^2$.

PROBLEM I.

To resolve a Pure Quadratic Equation, involving only one unknown Quantity.

RULE. Find the value of the square of the unknown quantity in exactly the same manner as we found the value of the simple quantity in Chapter XIII. Then extract the square root of each side according to the sixth General Rule.

NOTE. If the value of the square is a negative quantity, the extraction of its square root is impossible; but, if positive, it may have two square roots, one positive and the other negative. In the resolution of equations, we must attach both signs to the root, since the one answers as well as the other.

When the square of the unknown quantity is found equal to a negative quantity, we can have no real value of the unknown quantity. The statement conveyed in the given equation must have been an impossibility.

EXAMPLE. Given $2x^2 - 45 = 200 - 3x^2$.

Transposing, $5x^2 = 245$,

Dividing by 5, $x^2 = 49$.

Extracting the sq. root of both sides, $x = \pm 7$.

EXERCISE 1. Given $4x^2 - 85 = 90 - 3x^2$. Ans. $x = \pm 5$.

2. Given $\frac{x^2}{2} + \frac{x^2}{3} = \frac{x^2}{4} + 21$. Ans. $x = \pm 6$.

3. Given $\frac{6x^2-4}{3}-2=\frac{18-4x^2}{3}+x^2$. Ans. $x=\pm 2$.

4. Given $\sqrt{(x+1)}\times\sqrt{(x-1)}=x-\frac{1}{x}$. Ans. $x=\pm 1$.

PROBLEM II.

To resolve an Adfected Quadratic Equation, involving only one unknown Quantity.

RULE. Having simplified the equation, when necessary, by removing fractions, and by bringing all the terms containing only known quantities to the right side, and those containing the unknown quantity to the left, collect all the terms containing the square of the unknown quantity into one, and all those containing its first power into another. We shall then have the equation in the form $ax^2 \pm bx = \pm c$.

Next, if the co-efficient (a) of the first term is not 1, divide both sides of the equation by that co-efficient. The equation will then appear in the form $x^2+mx=n$, m and n being any known quantities, positive or negative.

To both sides add the square of half the co-efficient of the second term: the equation will then be of the form $x^2+mx+\frac{m^2}{4}=\frac{m^2}{4}+n$. This is called *completing the square*, since the left side of the equation is now a complete square.

Extract the square root of each side, and we have then the equation in the form $x+\frac{m}{2}=\sqrt{\left(\frac{m^2}{4}+n\right)}$, and consequently, $x=\sqrt{\left(\frac{m^2}{4}+n\right)}-\frac{m}{2}$ or $x=\frac{1}{2}(\sqrt{m^2+4n}-m)$.

NOTE 1. If n is a negative quantity and greater than $\frac{1}{4}m^2$, then $\sqrt{(m^2+4n)}$ is an impossible quantity, and no answer can be found to the question. It shows that the conditions of the proposed equation or question are inconsistent with each other. But if m^2+4n is a positive quantity, its square root may either be positive or negative; consequently x will have two values, either of which will satisfy the conditions of the given equation.

EXAMPLE 1. Given $x^2-8x=33$, to find x.

Completing the square by adding, to each side $\left(\frac{8}{2}\right)^2$, or 16, we have

$$x^2-8x+16=49.$$

Extracting the square root of each side,

$$x-4=\pm 7.$$

Consequently, $x=4\pm 7=+11$ or -3.

Exercise 1. Given $x^2+10x=75$.
Ans. $x=+5$ or -15.

2. Given $x^2-4x=12$. Ans. $x=+6$ or -2.

Example 2. Given $7x^2-63x+56=0$, to find x.

Transposing 56 and dividing by 7,

$$x^2-9x=-8.$$

Completing the square,

$$x^2-9x+\left(\frac{9}{2}\right)^2=\frac{81}{4}-8=\frac{49}{4}.$$

Extracting the square root,

$$x-\frac{9}{2}=\pm\frac{7}{2}.$$

$$\therefore x=\frac{9}{2}\pm\frac{7}{2}=8 \text{ or } 1.$$

Exercise 3. Given $5x^2-20x+15=0$.
Ans. $x=3$ or 1.

4. Given $x^2+5x-4=100$. Ans. $x=8$ or -13.

5. Given $11x^2-88x+110=209$. Ans. $x=9$ or -1.

6. Given $x^2-10x-100=0$.
Ans. $x=16{\cdot}18034-$, or $-6{\cdot}18034-$.

7. Given $y^2-11y+15=0$.
Ans. $y=9.4051+$, or $1{\cdot}5949-$.

8. Given $z : z+40 :: 10 : z-40$.
Ans. $z=57{\cdot}0156$ &c., or $-7{\cdot}0156$ &c.

9. Given $\frac{x^2}{12}+\frac{x}{2}=6$. Ans. $x=6$ or -12.

10. Given $\frac{1}{6}(z^2-z)=7$. Ans. $z=7$ or -6.

11. Given $\frac{3}{2}x^2-x=4$. Ans. $x=2$ or $-\frac{4}{3}$.

12. Given $\frac{x^2}{3}-\frac{2x}{5}=\frac{8}{15}$. Ans. $x=2$ or $-\frac{4}{5}$.

13. Given $\frac{y^2}{4}-\frac{y}{5}=7{\cdot}8$. Ans. $y=6$ or $-5{\cdot}2$.

14. Given $z^2 + 8z + 20 = 0$. Ans. $z = 4 \pm \sqrt{-4}$.

15. Given $\frac{x^2}{3} - \frac{x}{5} - 70 = 2$. Ans. $x = 15$ or $-14\frac{2}{5}$.

16. Given $\frac{125 - 3x}{x} - 4x = 2$. Ans. $x = 5$ or $-6\frac{1}{4}$.

17. Given $\frac{20}{x} - \frac{10x + 8}{x^2} - 2 = 0$. Ans. $x = 4$ or 1.

18. Given $2y - \frac{8y - 6}{y + 3} = 9$. Ans. $y = 7$ or $-\frac{3}{2}$.

19. Given $\sqrt[3]{(z^2 - 6z)} = -2$. Ans. $z = 4$ or 2.

20. Given $\sqrt{(x + 4)} \times \sqrt{(x - 3)} = 2x - 6$.
Ans. $x = 5\frac{1}{3}$ or 3.

21. Given $x + \sqrt{x} = 15$.
Ans. $x = 19{\cdot}405125 -$, or $11{\cdot}594875 +$.

Note 2. When the equation has been reduced to its simplest form, and the terms arranged according to the powers of the unknown quantity, it may be resolved as a quadratic although the highest power may not be the square, provided that the exponent of the power in the first term be double that in the second, and that the unknown quantity does not appear in any other term. In this case, however, we do not at once find the unknown quantity itself, but that power of it which appears in the second term; and from that,—by extracting the root indicated by its exponent, we obtain the first power.

Example 3. Given $z^6 - 10z^3 = 459$, to find z.

Completing the square, $z^6 - 10z^3 + 25 = 484$.
Extracting the sq. root, $z^3 - 5 = \pm 22$.
$\therefore z^3 = +27$ or -17.
Extracting the cube root, $z = +3$, or $-2{\cdot}57$ &c.

Exercise 22. Given $x^4 + 2x^2 = 10200$.
Ans. $x = 10$ or $\pm\sqrt{(-102)}$.

23. Given $x^6 - 35x^3 + 216 = 0$. Ans. $x = 2$ or 3.

24. Given $y - \sqrt{y} = 20$. Ans. $y = (+5)^2$ or $(-4)^2$.

Problem III.

To resolve a pure or adfected Quadratic Equation, involving two or more unknown Quantities.

Rule. Eliminate all the unknown quantities except one, *in the same* manner as in simple equations, and then find the value of the remaining unknown quantity by Problem I or II of this Chapter.

Note. The process will often require to be diversified according to circumstances, affording full scope for the exercise of ingenuity. Sometimes instead of first eliminating one of the unknown quantities, it will be better to form a complete square containing both the unknown quantities, or sometimes even two such complete squares, and then, on extracting the root, simplify one or both equations before the process of elimination commences.

It will frequently happen that equations, which appear to be simple, will assume the quadratic form when one or more unknown quantities are eliminated, and those which appear to be quadratic will rise to a higher order.

Example 1. Given $\left\{\begin{array}{l} x^2+y^2=85, \\ x^2-y^2=13, \end{array}\right\}$ to find x and y.

Taking successively the sum and difference of the two given equations, we have

$$2x^2=98, \text{ and } 2y^2=72.$$

$$\therefore x^2=49, \text{ and } y^2=36.$$

$$\text{Hence, } x=\pm 7, \text{ and } y=\pm 6.$$

Example 2. Given $\left\{\begin{array}{l} x^2-y^2=75, \\ \quad xy\ =50, \end{array}\right\}$ to find x and y.

Multiplying the first eq. by x^2 and squaring the second,

$$x^4-x^2y^2=75x^2,$$

$$\text{and } x^2y^2=2500.$$

Add these two equations together and transpose $75x^2$,

$$x^4-75x^2=2500.$$

Complete the square, and extract the sq. root,

$$x^2-\frac{75}{2}=\pm\frac{125}{2}.$$

$$\text{Hence, } x^2=+100 \text{ or } -25,$$

$$\text{and } x=\pm 10 \text{ or } \pm\sqrt{(-25)}.$$

$$\therefore y^2=25 \text{ or } -100, \text{ and } y=\pm 5 \text{ or } \pm\sqrt{-100}.$$

Example 3. Given $\left\{\begin{array}{l} x^2+y^2=125, \\ x\ +y\ =15, \end{array}\right\}$ to find x and y.

Squaring the second equation,

$$x^2+2xy+y^2=225.$$

Subtracting this from twice the first,

$$x^2-2xy+y^2=25.$$

Extracting the square root, $x-y=\pm 5$.

From this equation and the second, by taking their sum and difference, we have

$$2x=20 \text{ or } 10, \text{ and } 2y=10 \text{ or } 20.$$
$$\therefore\ x=10 \text{ or } 5, \text{ and } y=5 \text{ or } 10.$$

Exercise 1. Given $\begin{cases} 3x^2+4y^2=43, \\ 2x^2-3y^2=\ \ 6. \end{cases}$ Ans. $\begin{cases} x=\pm 3, \\ y=\pm 2. \end{cases}$

2. Given $\begin{cases} x+y : x-y :: 5 : 1, \\ 4 : x :: y : 24. \end{cases}$ Ans. $\begin{cases} x=\pm 12, \\ y=\pm\ \ 8. \end{cases}$

3. Given $\begin{cases} x \times z=9, \\ x \div z=4. \end{cases}$ Ans. $\begin{cases} x=\pm 6, \\ y=\pm 1\tfrac{1}{2}. \end{cases}$

4. Given $\begin{cases} x-3y=\ \ 9, \\ y^2-4y+2x=26. \end{cases}$ Ans. $\begin{cases} x=15 \text{ or } -3, \\ y=\ \ 2 \text{ or } -4. \end{cases}$

5. Given $\begin{cases} 5x^2-8y^2=+72, \\ 5x\ -8y\ =-12. \end{cases}$ Ans. $\begin{cases} x=12 \text{ or } -4, \\ y=\ \ 9 \text{ or } -1. \end{cases}$

6. Given $\begin{cases} x+y=34, \\ xy=273. \end{cases}$ Ans. $\begin{cases} x=13 \text{ or } 21, \\ y=21 \text{ or } 13. \end{cases}$

7. Given $\begin{cases} x^2+y^2=32{\cdot}5, \\ xy=15{\cdot}75. \end{cases}$ Ans. $\begin{cases} x=3{\cdot}5 \text{ or } 4{\cdot}5, \\ y=4{\cdot}5 \text{ or } 3{\cdot}5. \end{cases}$

8. Given $\begin{cases} x^2+y^2=8\tfrac{1}{2}, \\ x\ -y\ =1. \end{cases}$ Ans. $\begin{cases} x=2\tfrac{1}{2} \text{ or } -1\tfrac{1}{2}, \\ y=1\tfrac{1}{2} \text{ or } -2\tfrac{1}{2}. \end{cases}$

9. Given $\begin{cases} x+y=31{\cdot}25, \\ \sqrt{x}+\sqrt{y}=7{\cdot}5. \end{cases}$ Ans. $\begin{cases} x=25 \text{ or } 6\tfrac{1}{4}, \\ y=6\tfrac{1}{4} \text{ or } 25. \end{cases}$

10. Given $\begin{cases} x+z : x-z :: 11 : 3, \\ (x-5)\times(z-3)=45. \end{cases}$

Ans. $\begin{cases} x=14 \text{ or } -3\tfrac{3}{4}, \\ z=\ \ 8 \text{ or } -2\tfrac{1}{7}. \end{cases}$

11. Given $\begin{cases} x^3+y^3=35, \\ x\ +y\ =\ \ 5. \end{cases}$ Ans. $\begin{cases} x=3 \text{ or } 2, \\ y=2 \text{ or } 3. \end{cases}$

12. Given $\begin{cases} x+y+z=12, \\ x+2y+3z=26, \\ x^2+y^2+z^2=50. \end{cases}$ Ans. $\begin{cases} x=3. \\ y=4. \\ z=5. \end{cases}$

CHAPTER XVII.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

Note. When the question has been expressed in the form of a quadratic equation, and that equation resolved *by the rules in* the preceding Chapter, double values of the unknown quantities will be obtained, and each of these values *may* satisfy the conditions of the question (as in Example 1): but it is equally possible that one of the values may not be applicable as an answer to the question, although quite satisfactory as far as regards the equation. This may arise either from the negative form of the value obtained (as in Example 2), or from some other peculiar cause (as in Example 3).

The reason that the equation may be satisfied by the second value while the question is not, arises from the fact of some condition being stated or understood in the question, which has not been expressed, and probably could not be expressed, in the equation.

EXAMPLE 1. What number is that which exceeds the square of its half by $\frac{3}{4}$?

Let x be the number. Then

$$x-\left(\frac{x}{2}\right)^2=\frac{3}{4}.$$

This equation, resolved as usual, gives $x=3$ or 1, either of which numbers fulfils the conditions of the question.

EXAMPLE 2. A party of labourers were sent to remove a bank of earth containing 350 cubic yards: but, just as they were commencing, four of them were disabled by an accidental fall of part of the bank: in consequence of this it was found that the rest had each ten yards more to remove. What was the number of the party at first?

Let x express the number at first, and consequently $x-4$ the number after the accident. Then

$$\frac{350}{x}+10=\frac{350}{x-4}.$$

If we resolve this by the rules of the preceding Chapters, we find $x=14$ or -10. Either of these numbers would satisfy the equation, but only the first will be applicable to the question, since the supposition of a negative number of men being sent is ridiculous.

EXAMPLE 3. A number, expressed by two digits, is subtracted from its reverse. It is then found that the remainder is three fourths of the original number, but greater than the square of the smaller digit by 20. What is the number?

Let x be the right-hand digit and y the left, of which y

must be the smaller, otherwise the remainder would be negative. Then,

$$\begin{aligned} 10y + x &= \text{the number,} \\ 10x + y &= \text{the reverse,} \\ \hline \text{and } 9x - 9y &= \text{the remainder.} \end{aligned}$$

$$\begin{aligned} \text{Hence, } 9x - 9y &= \tfrac{3}{4}(10y + x), \\ \text{and } 9x - 9y &= y^2 + 20. \end{aligned}$$

From the former of these two equations we obtain $x = 2y$; and, inserting $2y$ for x in the latter, we find

$$y^2 - 9y = -20;$$

From which we find $y = 5$ or 4, and $\therefore x = 10$ or 8.

But of these two values the second only is applicable, since y cannot be 5 unless x be 10, which cannot be a digit. Therefore the number is 48.

Exercise 1. Divide the number 10 into two such parts that the sum of their squares shall be $54\frac{1}{2}$.
Ans. $6\frac{1}{2}$ and $3\frac{1}{2}$.

2. Divide 100 into two such parts that their product shall be 2211. Ans. 67 and 33.

3. There are two numbers in the proportion of 4 to 7, the sum of whose square is 4160. What are the numbers?
Ans. ± 32 and ± 56.

4. A person being asked how old he was, replied that the square of the number representing his age, in years, exceeded 20 times that number by 125. How old was he? Ans. 25.

5. Divide the number 21 into two such parts, that the squares of those parts may be in the proportion of 9 to 16.
Ans. 9 and 12.

6. Divide 100 into two such parts that their product may be equal to the difference of their squares.
Ans. 61·80340 −, and 38·19660 +.

7. A traveller, having proceeded 175 miles by railway, complained of the slowness of the train, and showed that, if it had only gone five miles per hour faster, he would have accomplished his journey in less time by an hour and three quarters. What was the rate at which the train went? Ans. 20 miles per hour.

8. What two numbers are those whose difference multiplied by the greater produces 104, and by the less, 88?
Ans. ± 26 and ± 22.

9. A toyman purchased a lot of cricket-balls at 28s. the dozen, and, after retailing them at a uniform price, found that he had received six guineas for the whole, and that the sum he received for the last dozen was all clear profit. How many did he sell, and at what price?

Ans. 36 at 3s. 6d., or 18 at 7s. each.

10. The difference of two numbers is 4, and their product 117. What are the numbers? Ans. ± 13, and ± 9.

11. The difference of two numbers is 2, and the difference of their cubes 152. What are the numbers?

Ans. ± 6, and ± 4.

12. A number of schoolboys, contributing equal sums, raised £6 for an exhibition of fire-works. Eight of their school-fellows, disapproving of the project, refused to contribute; but, had they joined, the others would have had sixpence a-piece less to pay in order to make up the sum. How many contributed? Ans. 48.

CHAPTER XVIII.

ARITHMETICAL PROGRESSIONS.

DEFINITION.

If a series of quantities increase or decrease by equal differences, it is called an *Arithmetical Progression*,—

as 7, 9, 11, 13, 15, 17;
or $16x$, $14x$, $12x$, $10x$, $8x$.

If a represent the first term, d the common difference, and n the number of terms, any arithmetical progression will be represented thus

$$a,\ a \pm d,\ a \pm 2d,\ a \pm 3d,\ \ldots\ a \pm (n-1)d.$$

Exercise. Fill up the blanks in the following arithmetical progressions:—

1, 2, —, —, —, —, 7;
2, 5, —, 11, —, —, —;
50, 45, —, —, —, —, —;
—, —, —, —, —, 36, 42;
x, $x + 2y$, ——, —, ——.

THEOREM.

In any arithmetical progression, the sum of the two extreme terms is equal to the sum of any two terms equidistant from the extremes, or to double the middle term when the number of terms is odd.

NOTE. Any *part* of an arithmetical progression is, of course, an arithmetical progression.

EXERCISE 1. Fill up the blanks in the progression, 7, —, —, —, 19.

2. What is the middle term of an arithmetical progression of eleven terms, having its first term $156a$, and its last term $246a+10d$? Ans. $201a+5d$.

PROBLEM I.

Having given the common Difference of an arithmetical progression and the first Term, to find any other Term (say the nth).

RULE. Multiply the common difference by 1 less than the number of the term required (*i. e.* by $n-1$), and add the product to the first term, or subtract it from it, according as the progression increases or decreases.

EXAMPLE. What is the twentieth term of the progression $3a$, $5a$, $7a$, &c.?

$$3a + 2a \times 19 = 41a. \text{ Ans.}$$

EXERCISE 1. What is the fiftieth term of the progression, 1, 3, 5, &c.? Ans. 99.

2. What is the tenth term of the progression $30c$, $27c$, $24c$, &c.? Ans. $3c$.

3. What is the twelfth term of the series a, $a+5$, $a+10$, &c.? Ans. $a+55$.

4. Compute the sixteenth term of the progression, x, $x-2y$, $x-4y$, &c. Ans. $x-30y$.

PROBLEM II.

To find the Sum of an arithmetical Progression.

RULE. Multiply half the sum of the first term and *the last by the* number of terms.

Exercises. Sum the following progressions :—

1. $1, 2, 3, 4, \ldots \ldots 100.$ Ans. 5050.

2. $17c, 15c, 13c, 11c, 9c, 7c.$ Ans. $72c.$

3. $a, a+b, a+2b, \ldots \ldots a+10b.$ Ans. $11a+55b.$

Exercise 4. Find the sum of 21 terms of the series, $x, \frac{3x}{2}, 2x, \frac{5x}{2}$, &c. Ans. $126x.$

5. A young man, engaging as a farm servant, asked ten pounds a year as his wage. His employer refused to give so much at first, but offered him 5 guineas for the first year, to be raised 15 shillings every year that he remained in his place. He remained 40 years on these terms. What was his last year's wage, and what did he earn during the whole time? Ans. £34, 10s., and £795.

CHAPTER XIX.

GEOMETRICAL PROGRESSIONS.

DEFINITIONS.

If a series of quantities increase from continual multiplication by the same quantity, or decrease from continual division by the same quantity, the series is called a *Geometrical Progression*,—

as, 2, 6, 18, 54, 162, 486;
or 64, 32, 16, 8, 4, 2.

Even when the series decreases it may be considered to do so by multiplication, the multiplier, in that case, being a fraction. Consequently every geometrical progression may be expressed thus,—

$$a, ar, ar^2, ar^3, \ldots \ldots \ldots ar^{n-1}.$$

Regarding all geometrical progressions in the latter point of view, viz. as formed by successive multiplications, the constant multiplier, r, is called the *common Ratio* of the terms. As usual n expresses the number of terms.

Exercise 1. What are the values of r and n in each of the two numerical progressions given above?

2. Form a geometrical progression of five terms, having its first term $7a$, and the common ratio of the terms 2.

3. Form a geometrical progression of four terms, having its first term $4x$, and the common ratio $5y$.

4. The first term of a geometrical progression is y, the second $y \div 3$, and the last z. What is the last but one? Ans. $3z$.

5. If the second and third terms are $\frac{x}{y}$ and $\frac{y}{x}$, what are the first and fourth? Ans. $\frac{x^3}{y^3}$, and $\frac{y^3}{x^3}$.

NOTE. The successive terms of a geometrical progression are continual proportionals; that is,—the first is to the second as the second to the third, as the third to the fourth, and so on. Consequently any term, standing between two others, is a mean proportional between them.

EXERCISE 3. Fill up the blanks in the following geometrical progressions:—

3, —, 12, —, —, —.
—, —, $9a$, —, $81a$.

THEOREM.

In any geometrical progression the product of the two extreme terms is equal to the product of any two terms equidistant from them; and also equal to the square of the middle term when the number of terms is odd.

EXERCISE 1. In a geometrical progression consisting of seven terms the first is 13, and the middle one 104. What is the last? Ans. 832.

2. If the first term is $7a$, and the fifth $567a^5b^4$, what are the other terms? Ans. $21a^2b$, $63a^3b^2$, and $189a^4b^3$.

PROBLEM I.

Having given the first Term of a geometrical Progression, and the common Ratio, to find any other Term (say the nth).

RULE. Raise the common ratio to a power whose exponent is a unit less than the number of the term required, *viz.* $n - 1$, and multiply that power into the first term.

Example. The first term of a geometrical progression being 50, and the common ratio 7, what is the fourth term?

$$7^3 \times 50 = 343 \times 50 = 17150. \text{ Ans.}$$

Exercise 1. The first term of a geometrical progression is 13; and the common ratio, 2. What is the tenth term?
Ans. 6656.

2. The common ratio being $\frac{a}{3}$ and the first term $\frac{27}{a^3}$, what is the seventh term? Ans. $\frac{a^3}{27}$.

Problem II.

Having given the first Term of a geometrical Progression, the last Term, and the common Ratio, to find the Sum of all the Terms.

Rule. Divide the difference of the first term and the last by the difference of the common ratio and unity. To the quotient add the last term.

Example. The first term of a geometrical progression is a, and the common ratio is r: what is the sum of n terms?

By Pr. I., the nth term is ar^{n-1}.

$$\therefore \text{ Sum} = \frac{ar^{n-1} - a}{r-1} + ar^{n-1} = \frac{ar^n - a}{r-1}.$$

Exercise 1. The first term of a geometrical progression is 3, and the common ratio 2: what is the sum of ten terms? Ans. 3069.

2. What is the sum of a geometrical progression of eight terms, whose first term is 413,343, and common ratio, $\frac{1}{3}$? Ans. 619,920.

Note. By the same rule we may find the sum of a decreasing geometrical progression, even when the number of terms is infinite, if we regard the last term as $= 0$.

Exercise 3. What is the sum of the series, $1 + \frac{1}{2} + \frac{1}{4} +$ &c., *ad infinitum*? Ans. 2.

4. What is the sum of the series, $1 + \frac{2}{3} + \frac{4}{9} +$ &c., *ad infinitum*? Ans. 3.

CHAPTER XX.

UNRESOLVED EXERCISES.

Exercises in Chapter II, Problem I.

Sum each of the following groups of quantities :—

(1.)
$$\begin{array}{r} 15ab \\ 7ab \\ 32ab \\ ab \\ 55ab \\ 93ab. \\ \hline \end{array}$$

(2.)
$$\begin{array}{r} +84\sqrt{x} \\ +29\sqrt{x} \\ +5\sqrt{x} \\ +48\sqrt{x} \\ +16\sqrt{x} \\ +57\sqrt{x}. \\ \hline \end{array}$$

(3.)
$$\begin{array}{r} -208x^2 \\ -36x^2 \\ -57x^2 \\ -726x^2 \\ -244x^2 \\ -862x^2. \\ \hline \end{array}$$

(4.)
$$\begin{array}{r} -377\times\frac{y}{z} \\ -800\times\frac{y}{z} \\ -493\times\frac{y}{z}. \\ \hline \end{array}$$

(5.)
$$\begin{array}{r} +55c \\ -43c. \\ \hline \end{array}$$

(6.)
$$\begin{array}{r} +12d^2 \\ -96d^2. \\ \hline \end{array}$$

(7.)
$$\begin{array}{r} -499\sqrt{xy} \\ +267\sqrt{xy}. \\ \hline \end{array}$$

(8.)
$$\begin{array}{r} -54az \\ +300az. \\ \hline \end{array}$$

(9.)
$$\begin{array}{r} +8y^2 \\ -7y^2 \\ +y^2 \\ -5y^2 \\ -9y^2 \\ -4y^2. \\ \hline \end{array}$$

(10.)
$$\begin{array}{r} -540abc \\ +365abc \\ +286abc \\ +194abc \\ -868abc \\ +444abc. \\ \hline \end{array}$$

(11.)
$$\begin{array}{r} -25a^2x^2 \\ -14a^2x^2 \\ +62a^2x^2 \\ -49a^2x^2 \\ +72a^2x^2 \\ +50a^2x^2. \\ \hline \end{array}$$

(12.)
$$\begin{array}{r} +430\sqrt{z} \\ +28\sqrt{z} \\ +196\sqrt{z} \\ -85\sqrt{z} \\ -243\sqrt{z} \\ -9\sqrt{z}. \\ \hline \end{array}$$

(13.)
$$\begin{array}{r} 63x \\ 38y \\ 54z \\ 20y \\ 92z \\ 13z. \\ \hline \end{array}$$

(14.)
$$\begin{array}{r} +12a \\ -19b \\ +48a \\ +63b \\ -15b \\ -8a. \\ \hline \end{array}$$

(15.)
$$\begin{array}{r} -308\sqrt{ax^3} \\ +122\sqrt{bx^3} \\ +424\sqrt{ax^3} \\ -852\sqrt{bx^3} \\ +66\sqrt{ax^3} \\ -508\sqrt{bx^3}. \\ \hline \end{array}$$

(16.)
$$\begin{array}{r} +17mx \\ -19nx \\ +34px \\ +82px \\ -54mx \\ -80mx. \\ \hline \end{array}$$

EXERCISES IN CHAPTER II, PROBLEM II.

(1.)	(2.)	(3.)
$28x-28y$	$+33a-44b$	$+\ 8\sqrt{x}-9\sqrt{y}$
$15x-14y$	$-60a-18b$	$+12\sqrt{x}+7\sqrt{y}$
$34x+50y$	$+55a-27b$	$-87\sqrt{x}-5\sqrt{y}$
$17x-12y$	$-83a-83b$	$-34\sqrt{x}+8\sqrt{y}$
$86x-42y$	$-24a-36b$	$+90\sqrt{x}+\ \sqrt{y}$
$17x+74y$.	$+72a-\ 8b$.	$-38\sqrt{x}-6\sqrt{y}$.

(4.) Add together $a^2+b^2+c^2+d^2$, $a^2-2ab+b^2$, $a^2+10ac+25c^2$, $b^2-6bc+9c^2$, $4c^2-4cd+d^2$, $9a^2-24ad+16d^2$, $-16b^2+24bd-9d^2$, and $10ab+10ac+10bc+10ad+10bd+10cd$.

(5.) $+8(a^2-x^2)$, $-19(a^2-x^2)$, $+86(a^2-x^2)$, and $-27(a^2-x^2)$.

(6.) $+38(b+c)^3$, $-54(b+c)^3$, $-27(b+c)^3$, $-93(b+c)^3$, and $+804(b+c)^3$.

(7.) $+59\times\dfrac{ax+by}{xy}$, $+94\times\dfrac{ax+by}{xy}$, and $-35\times\dfrac{ax+by}{xy}$.

(8.) $+12(ab+cd)$, $-22(ac+bd)$, $-5(ab+cd)$, and $+77(ac+bd)$.

EXERCISES IN CHAPTER III.

(1.) From $5800x^2y^2$ subtract $2438x^2y^2$.
(2.) From $-34abc$ subtract $-100abc$.
(3.) From $+17my$ subtract $+83my$.
(4.) From $-88\sqrt{x^5}$ subtract $-29\sqrt{x^5}$.
(5.) From $81\sqrt{ay}-12\sqrt{bz}$ take $29\sqrt{ay}-20\sqrt{bz}$.
(6.) From $267mx^2-969ny^2$ take $607mx^2-242ny^2$.
(7.) From $9a-3b+2c-8d$ take $-3a+8b-7c+5d$.
(8.) Take $584x-362y+607z$ from $372x-584y-43z$.
(9.) From $45(a+x)-63(a+y)$ take $24(a+x)-86(a+y)$.

EXERCISES IN CHAPTER IV.

Simplify the following expressions :—
(1.) $(a+b)+(a-b)-(a-c)-(b+c)$.
(2.) $(7w+35x-54y-12z)-(62w-12x+13y-80z)$.

Exercises in Chapter V, Problem II.

(1.) Multiply $35a^3 + 12a^2x - 48ax^2 - 92x^3$ by a^2x^2.
(2.) Multiply $612b^5y^7 - 356b^3y^9$ by $-369a^2b^5y$.

Exercises in Chapter V, Problem III.

(1.) Multiply $3a - 5b$ by $7a - 6b$.
(2.) Multiply $34c - 35d$ by $35c - 34d$.
(3.) Multiply $7m - 9n$ by $9n - 7m$.
(4.) Multiply $a^2 + 2ab + b^2$ by $a^2 - 2ab + b^2$.
(5.) Multiply $a - 3a^2 + 9a^3$ into $1 + 3a$.
(6.) Multiply $27x^3 + 27x^2y + 9xy^2 + y^3$ by $27x^3 - 27x^2y + 9xy^2 - y^3$.

(7.) Multiply $54(x - y)$ into $75(x - y)$.
(8.) Multiply $-43(y + z)^3$ into $-26(y + z)^5$.
(9.) Multiply $5(a + b) + 7(a - b)$ by $5(a + b) - 7(a - b)$.

Exercises in Chapter V, Problem IV.

(1.) Find the continued product of $+3a^2x$, $-5b^2y$, $-7bx^2$, and $+4ay^2$.
(2.) Of $a + b$, $a + c$, and $a + d$.
(3.) Of $x - 2$, $x - 3$, $x - 4$, and $x - 5$.
(4.) Of $x + y + z$, $x + y - z$, $x - y + z$, and $-x + y + z$.
(5.) Of $a^2 + 7a + 49$, $a^2 - 7a + 49$, $a + 7$, and $a - 7$.

Exercises in Chapter VI, Problem I.

(1.) Divide $+1728a^5b^4c^3$ by $-144a^3b^3c^3$.
(2.) Divide $-64x^4y^4z^4$ by $-56xy^2z^3$.
(3.) Divide $-1080xy^3$ by $+45x^3y$.

Exercises in Chapter VI, Problem II.

(1.) Divide $35a^4b - 84a^3b^2 - 105a^2b^3$ by $7a^2b$.
(2.) Divide $96m^2a^2 - 9m^2n^2$ by $-12mn$.

EXERCISES IN CHAPTER VI, PROBLEM III.

(1.) Divide $b^2 + 4by + 4y^2$ by $b + 2y$.
(2.) Divide $8a^2 - 2ab - 15b^2$ by $4a + 5b$.
(3.) Divide $-16x^4 + 9y^4$ by $4x^2 - 3y^2$.
(4.) Divide $a^3 - b^3$ by $a - b$.
(5.) Divide $25a^4 + 50a^2 - 100a + 200$ by $5a^2 - 10a + 10$.
(6.) Divide $z^5 - 3125$ by $z - 5$.

(7.) Divide $10a^2 + 49ab - 10b^2$ by $2a + 11b$.
(8.) Divide $25x^4 - 90z^4$ by $5x^2 - 9z^2$.
(9.) Divide $84a^4 + 99a^3 - 7a^2 + 90a - 81$ by $12a^2 - 3a + 5$.
(10.) Divide $8x^3 - 4x^2y - xy^2$ by $4x^2 + 4xy + y^2$.
(11.) Divide $30y^3 - 67y^2 + 80y$ by $5y^2 - 7y + 6$.

(12.) Divide $x^5 - 4x^3 - 20x$ by $x^3 - 2x^2 + 3x - 4$.
(13.) Divide $3a^4 - a^3b - 14a^2b^2$ by $3a^3 + 5a^2b - 2ab^2 + b^3$.

(14.) Divide $2 + 6x$ by $1 - 3x$.
(15.) Divide 10 by $1 - 5y$.

EXERCISES IN CHAPTER VII.

(1.) Find the greatest common measure of $20{,}952ab^3x^5$, and $12{,}528a^5b^2x^2$.
(2.) Of $125x^3 + 64y^3$, and $25x^2 - 16y^2$.
(3.) Of $729a^6 - 243a^4 + 27a^2 - 2$, and $81a^4 - 27a^2 + 2$.

(4.) Of $16a^4 - 12a^2 + 2$, and $64a^6 - 48a^4 + 12a^2 - 2$.
(5.) Of $18x^3 - 53xy^2 - 28y^3$, and $36x^4 - 48x^3y - 56xy^3 - 49y^4$.

(6.) Of $12a^5 + 32a^4b - 12a^3b^2$, and $8a^4 - 72a^2b^2$.
(7.) Of $6y^4 - 10y^3 + 32y$, and $10y^4 - 10y^3 - 20y^2 + 80y$.

EXERCISES IN CHAPTER VIII, PROBLEM I.

(1.) Find the least common multiple of 144 and 96.
(2.) Of $11a$ and $17a$.

(3.) Of $48x^2y$ and $60xy^2$.
(4.) Of $3a^2+12ab$ and $5ab+20b^2$.
(5.) Of $8b^3-12b^2c+18bc^2-27c^3$ and $8b^3+12b^2c+18bc^2+27c^3$.

EXERCISES IN CHAPTER VIII, PROBLEM II.

(1.) Find the least common multiple of 14, 21, and 24.
(2.) Of $x^3y^2z^2$, $x^2y^2z^3$, and $x^2y^3z^2$.
(3.) Of 48, 60, 35, and 42.
(4.) Of $8a$, $24a$, $18a$, $54a$, $9a$, $15a$, and $27a$.
(5.) Of $5a$, $6b$, $7c$, and $11d$.
(6.) Of $9a^2-16b^2$, $9a^2+24ab+16b^2$, and $9a^2-24ab+16b^2$.

EXERCISES IN CHAPTER IX, PROBLEM I.

(1.) Reduce $15z$ to a fraction having its denominator 36.

(2.) Reduce $a-x$ to a fraction having its denominator $a+x$.

(3.) Reduce $y^2+7y+49$ to a fraction having its denominator $y-7$.

EXERCISES IN CHAPTER IX, PROBLEM II.

Reduce the following mixed numbers to fractional forms :—

(1.) $3a-\dfrac{7x^2}{5z}$.

(2.) $7a-14b+\dfrac{5a^2+3b^2}{8a}$.

(3.) $8x+\dfrac{11x^2-16xy+11y^2}{6y}$.

(4.) $7y-9z-\dfrac{49y^2+81z^2}{7y+9z}$.

(5.) $x^2-9-\dfrac{x^6-729}{x^4+9x^2+81}$.

EXERCISES IN CHAPTER IX, PROBLEM III.

Change the following fractions into integral or mixed quantities.

(1.) $\frac{36x^2y^5 - 45x^5y^2}{9x^2y^2.}$.

(2.) $\frac{24a^2 - 40b^2}{6a - 12b}$.

(3.) $\frac{8a^3 + 27b^3}{2a + 3b}$.

(4.) $\frac{48y^2 - 46yz}{6y - 11z}$.

(5.) $\frac{24x^3 - 13x^2}{8x^2 + 9x - 10}$.

(6.) $\frac{a^2 - 4b^2}{a + 2b - 3c}$.

EXERCISES IN CHAPTER IX, PROBLEM IV.

(1.) Change $\frac{3y^2 - 5z^2}{m}$ into a fraction having its denominator $2m^2n^2$.

(2.) Change $\frac{7a^2 - 4b^2}{x}$ into a fraction having its denominator $2ax + 3bx$.

(3.) Convert $\frac{a+x}{a-x}$ into an equivalent fraction having its denominator $(a-x)^2$.

EXERCISES IN CHAPTER IX, PROBLEM V.

Reduce the following fractions to their lowest terms:—

(1.) $\frac{888x^5y^3z}{925xy^3z^5}$.

(2.) $\frac{4a^3 + a^2b - 5ab^2}{4a^2 + 9ab + 5b^2}$.

(3.) $\frac{10y^3+22y^2-24y}{10y^3-38y^2+24y}$.

(4.) $\frac{12x^2-17xy-5y^2}{9x^3+21x^2y-60xy^2}$.

EXERCISES IN CHAPTER IX, PROBLEM VI.

(1.) Reduce $\frac{a}{2}$, $\frac{2b}{3}$, and $\frac{3c}{4}$ to equivalent fractions having the same denominator.

(2.) $\frac{2a}{15x^2y}$, and $\frac{3a}{20xy^2}$.

(3.) $\frac{5x}{12m^2}$, $\frac{7x}{20mn}$, and $\frac{8x}{21n^2}$.

(4.) $\frac{3a+5b}{18x}$, $\frac{8a-10b}{27y}$, and $\frac{9a^2-6b^2}{54xy}$.

(5.) $\frac{x+6}{x-6}$, and $\frac{x-6}{x+6}$.

(6.) $\frac{a^2+10x+25}{a^2-10x+25}$, $\frac{a^2+25}{a^2-25}$, and $\frac{a^2-10x+25}{a^2+10x+25}$.

EXERCISES IN CHAPTER IX, PROBLEM VII.

(1.) Add together $\frac{x^2+3y^2}{z}$, $\frac{3x^2+y^2}{z}$, and $\frac{2x^2-2y^2}{z}$,

(2.) $\frac{c}{2}$, $\frac{2c}{3}$, $\frac{3c}{4}$, and $\frac{4c}{5}$.

(3.) $\frac{3a}{8bc}$, $\frac{5b}{6ac}$, and $\frac{8c}{9ab}$.

(4.) $\frac{a}{a+b}$, and $\frac{a}{a-b}$.

(5.) $\frac{2a^2-3b^2}{6a^2-8ab}$, and $\frac{2a^2+3b^2}{15ab-20b^2}$.

(6.) $\frac{x^2-14x+49}{x^2+14x+49}$, $\frac{x^2+49}{x^2-49}$, and $\frac{x^2+14x+49}{x^2-14x+49}$.

(7.) $+\dfrac{1}{2a^2+2ax}, \quad +\dfrac{1}{2a^2-2ax},$ and $-\dfrac{1}{a^2+x^2}.$

(8.) $+\dfrac{a+b}{b}, \quad -\dfrac{a}{a+b},$ and $-\dfrac{a^3-a^2b}{a^2b-b^3}.$

Exercises in Chapter IX, Problem VIII.

(1.) From $\dfrac{15a+14b}{20}$ subtract $\dfrac{9a+8b}{12}.$

(2.) From $\dfrac{35a^3-48a^2+20a}{2x}$ subtract $\dfrac{30a^3-72a^2-5a}{3x}.$

(3.) From $\dfrac{7}{5a-4b}$ subtract $\dfrac{7}{5a+4b}.$

(4.) From $\dfrac{4a^2+9x^2}{4a^2-9x^2}$ subtract $\dfrac{4a^2-9x^2}{(2a+3x)^2}.$

Exercises in Chapter IX, Problem IX.

Find the following products :—

(1.) $\dfrac{x-7}{x-3}\times\dfrac{x+7}{x+3}\times\dfrac{x^2+49}{x^2+9}.$

(2.) $\dfrac{4x^2}{5y}\times\dfrac{15y^2}{16z}\times\dfrac{36z^2}{35x}.$

(3.) $\dfrac{2a-5}{3a-4}\times\dfrac{2a+5}{3a+4}\times\dfrac{9a^2-16}{4a^2-25}.$

(4.) $\dfrac{10ax-18ay}{7ab-14b^2}\times\dfrac{8a^2-16ab}{15bx-27by}.$

(5.) $\dfrac{a+b}{a^2-2ab+b^2}\times\dfrac{a^2+b^2}{a^2-b^2}\times(a-b)^3.$

(6.) $\dfrac{13a^2+15b^2}{13a^2-15b^2}\times 13a^2-15b^2.$

(7.) $\left(\frac{5a}{12}+\frac{3a}{14}-\frac{16a}{21}\right)\times 84.$

(8.) $\left(\frac{2x}{15}-\frac{3y}{20}+\frac{7z}{30}\right)\times 60.$

(9.) $\frac{a-b}{13}\times 39\,(a+b).$

(10.) $\left(11x+\frac{2y}{3}\right)\times\left(11x-\frac{2y}{3}\right).$

(11.) $\left(a+\frac{b}{c}\right)\times\left(a-\frac{c}{d}\right).$

EXERCISES IN CHAPTER IX, PROBLEM X.

(1.) Divide $\frac{36a^5b^2}{35xy}$ by $\frac{54a^3b^3}{49x^2y^2}.$

(2.) Divide $\frac{45x^2}{a^2-b^2}$ by $\frac{93y^2}{a^2+2ab+b^2}.$

(3.) Divide $\frac{15}{2x}-\frac{20}{3y}+\frac{30}{7z}$ by 60.

(4.) Divide $\frac{x^2+3x-10}{x^2-x-6}$ by $\frac{x^2+x-20}{x^2+x-12}.$

(5.) Divide $a^2-\frac{a^3}{b}$ by $\frac{a}{b}.$

(6.) Divide $64x^4-\frac{4y^2}{9}$ by $8x^2+\frac{2y}{3}.$

EXERCISES IN CHAPTER X, PROBLEM I.

(1.) Find the fifth power of $-7a^3b^5c^7$.
(2.) The ninth power of $-10x^6y^8$.

EXERCISES IN CHAPTER X, PROBLEM II.

(1.) Find the square of $5a-6b$.
(2.) The square of $7x-8y+9z$.

(3.) The cube of $a^2 - b^2$.
(4.) The fourth power of $3c - 4d$.
(5.) The fifth power of $x - y - z$.
(6.) The square of $6\sqrt{x} - 5\sqrt{y}$.
(7.) The square of $\frac{3x}{4} - \frac{4y}{3}$.
(8.) The cube of $2a - \frac{1}{2}b$.

EXERCISES IN CHAPTER X, PROBLEM III.

(1.) Find the square of $\frac{12x - 15y}{16x - 10y}$.
(2.) Find the fourth power of $\frac{5+z}{3-z}$.

EXERCISES IN CHAPTER XI, PROBLEM I.

(1.) Find the square root of $27{\cdot}04a^2b^4c^6$.
(2.) The cube root of $-216x^3y^9z^{12}$.
(3.) The fourth root of $4096c^{12}z^8$.
(4.) The square root of $-715ab^3$.
(5.) The cube root of $\frac{8a^3b^6}{27x^3}$.

EXERCISES IN CHAPTER XI, PROBLEM II.

(1.) Compute the square root of $16b^2 - 24bc + 9c^2$.
(2.) Of $64a^2x^2 - 96a^2xy + 144a^2y^2$.
(3.) Of $49a^2 - 126ab + 154ac + 81b^2 - 198bc + 121c^2$.
(4.) Of $\frac{4a^6 - 6a^3b^3 + 9b^6}{a^2 + 2ab + b^2}$.
(5.) Of $\frac{4}{9}a^2 - ab + \frac{9}{16}b^2$.

EXERCISES IN CHAPTER XII, PROBLEM I.

Supply the blank term in each of the following proportions:—

(1.) $14a^2bx$: ——— :: $21ax^2y$: $27bxy^2$.
(2.) $3a^2 + 3ax$: $3a^2 - 3ax$ :: ——— : $4ax - 4x^2$.

(3.) $\rule{1.5cm}{0.4pt} : \frac{x}{y} :: \frac{y}{x} : \frac{b}{a}$.

(4.) $4a^2-9b^2 : 6a^2-5ab-6b^2 :: 6a^2+5ab-6b^2 : \rule{1.5cm}{0.4pt}$.

(5.) Two quantities are to each other as 5 to 6. If the former be $150x$, what will the latter be?

(6.) A, working alone, can do a piece of work in 28 days, and B, working alone, can do it in 21 days. How long will it take them working together?

(7.) Three pumps are occasionally employed for filling a certain reservoir with water. The first and second, when worked together, can fill the cistern in 70 minutes; the first and third can fill it in 105 minutes; and the second and third, in 140 minutes. In what time can the reservoir be filled by each of the three pumps, when worked alone?

EXERCISES IN CHAPTER XII, PROBLEM II.

(1.) Find a mean proportional between 49 and 64.
(2.) Between $9a^2$ and $25b^2$.
(3.) Between $4a^2+12ab+9b^2$ and $9a^2+12ab+4b^2$.

EXERCISES IN CHAPTER XIV, PROBLEM I.

(1.) Given $x-59=75$.
(2.) ... $y+48=94$.
(3.) ... $654-z=308$.
(4.) ... $286y=2574$.
(5.) ... $93x=1441\frac{1}{2}$.
(6.) ... $x\div 13=16$.
(7.) ... $64z\div 25=320$.
(8.) ... $12y+74=20y-26$.
(9.) ... $\frac{3x}{5}-6a=15b$.
(10.) ... $\frac{7x-10}{23}+x=\frac{109-5x}{23}+7$.
(11.) ... $\frac{2x}{3}-\frac{3x}{4}+x=11$.
(12.) ... $\frac{2x}{5}+\frac{x}{6}-\frac{3x}{7}=10$.
(13.) ... $\frac{7x}{8}-\frac{5x}{6}=1$.

(14.) Given $\frac{x}{12}+\frac{x}{15}-\frac{x}{20}=9$.

(15.) ... $\frac{5y-5}{12}+\frac{3y+1}{10}=\frac{y}{2}+2\frac{1}{2}$.

(16.) ... $\frac{1}{5-x}-\frac{1}{5+x}=\frac{2}{25-x^2}$.

(17.) ... $\frac{34y-5}{20}+\frac{18y+3}{16}=\frac{42y}{15}$.

(18.) ... $\frac{z-3\frac{1}{2}}{2}-\frac{z+3\frac{1}{2}}{3}+\frac{z}{9}=0$.

(19.) ... $\frac{22-3z}{6}-\frac{76-9z}{14}=-\frac{6z-11}{21}$.

(20.) ... $\frac{5x-1}{2}-\frac{3x+15}{8}=6x-\frac{7x-3}{6}-11$.

(21.) ... $9\sqrt{18x}=108$.

(22.) ... $3\sqrt{(15x+4)}=21$.

(23.) ... $2\sqrt{(5y+11)}=\sqrt{(25y-66)}$.

(24.) ... $\sqrt{(x+5)}.\sqrt{(x-5)}=x-1$.

(25.) ... $\sqrt{\frac{y}{4}}+12=\sqrt{4y}$.

(26.) ... $\sqrt{(3x+31)}.\sqrt{(3x-9)}=3x+3$.

(27.) ... $\frac{4x-25}{2\sqrt{x}+5}=\sqrt{x}-2$.

(28.) ... $y-17 : y+17 :: 1 : 2$.

(29.) ... $\frac{x+5}{3} : \frac{21-x}{2} :: 2 : 3\frac{1}{2}$.

(30.) ... $\frac{4}{z-3} : \frac{5}{z-2} :: a : \frac{3a}{4}$.

(31.) ... $\sqrt{(x+5)} : \sqrt{(x-2)} :: \frac{2}{3} : \frac{1}{2}$.

(32.) ... $x^2-49 : x :: 2x+14 : 3$.

(33.) ... $5x+3 : 9x-3 :: 7x-5 : 8x+5$.

Exercises in Chapter XIV, Problem II.

(1.) Given $\begin{cases} 2x+3y=33, \\ 2x-3y=\ \ 3. \end{cases}$

(2.) ... $\begin{cases} 4x+5y=128, \\ 2x+3y=\ \ 72. \end{cases}$

(3.) Given $\begin{cases} 11y-4z = 76, \\ 15y-4z = 108. \end{cases}$

(4.) ... $\begin{cases} 6x-5z = 12, \\ 4x+3z = 15. \end{cases}$

(5.) ... $\begin{cases} 7x-2z = 134, \\ 5x-11z = 0. \end{cases}$

(6.) ... $\begin{cases} 16x-3y = 32, \\ 20x+6y = 53. \end{cases}$

(7.) ... $\begin{cases} 7x+y = 71, \\ 8x+2y = 112. \end{cases}$

(8.) ... $\begin{cases} 2\frac{1}{2}x + 3\frac{1}{3}y = 325, \\ 3\frac{1}{3}x + 2\frac{1}{2}y = 305. \end{cases}$

(9.) ... $\begin{cases} y-z = 2\frac{1}{4}, \\ y \div z = 2\frac{1}{2}. \end{cases}$

(10.) ... $\begin{cases} x^2-z^2 = 95, \\ x+z = 19. \end{cases}$

(11.) ... $\begin{cases} \dfrac{7x-9y}{3} = 8, \\ \dfrac{5x+5y}{4} = 16. \end{cases}$

(12.) ... $\begin{cases} \dfrac{x+y}{2} + \dfrac{x-y}{3} = 7, \\ \dfrac{2x+3y}{4} - \dfrac{3x+2y}{6} = \dfrac{x}{4}. \end{cases}$

(13.) ... $\begin{cases} \dfrac{3x+4z}{30} - \dfrac{5x+2z}{40} = 8\frac{4}{5}, \\ \dfrac{4x-z}{14} - \dfrac{2x-3z}{4} = 70\frac{1}{14}. \end{cases}$

(14.) ... $\begin{cases} \dfrac{11x-y+5}{9} - \dfrac{4(y-2)}{15} = \dfrac{3(x+y+4)}{10} + 28. \\ \dfrac{x+6}{10} : \dfrac{y-2}{15} :: 7\frac{1}{2} : 2\frac{1}{2}. \end{cases}$

EXERCISES IN CHAPTER XIV, PROBLEM III.

(1.) Given $\begin{cases} x+y = 5, \\ x+z = 6, \\ y+z = 8. \end{cases}$

(2.) Given $\begin{cases} x + y + z = 19, \\ x - y - z = 5, \\ x + 2y + 3z = 28. \end{cases}$

(3.) ... $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 10, \\ x + y + z = 30, \\ 2x + 3y + 4z = 100. \end{cases}$

(4.) ... $\begin{cases} 5x - 4y + 2z = +21, \\ 3x + 2y - 2z = +17, \\ 7x - 8y - 6z = -89. \end{cases}$

(5.) ... $\begin{cases} \frac{1}{5}(x + y) - \frac{1}{8}z = 2\frac{1}{2}, \\ \frac{1}{9}(y + z) - \frac{1}{6}x = 2\frac{1}{2}, \\ \frac{1}{3}(x + z) - \frac{1}{4}y = 5\frac{2}{3}. \end{cases}$

EXERCISES IN CHAPTER XV.

1. What number is that which is as much short of 100 as its double exceeds 100?

2. Divide 34 into two parts which shall be to each other in the proportion of $\frac{2}{3}$ to $\frac{3}{4}$.

3. Two tradesmen began business with equal sums of money. The first gained £142, and the second lost £150. The first was then twice as rich as the second. What sum had each at the commencement?

4. Four persons engaged in a speculation requiring an outlay of £2400. Of that sum, *A* contributed twice as much as *B*, and £10 more; *C* contributed £20 less than *A*; and *D* only two-thirds of *C*. What sums did they separately contribute?

5. Divide 100 into two such parts that 25 times the less may exceed 24 times the greater by 1.

6. Two fields were purchased at £43 and £28 per acre respectively. Their united area was 12 acres, and their united price £411. What were the sizes of the fields separately?

7. Two friends, living at Walton and Middleton, 24 miles apart, agreed to meet on an angling excursion between the two places. The one from Walton set out an hour after the other, but, having no encumbrance, got on at the rate of 4 miles an hour,—while his friend, having undertaken to bring fishing-tackle and provisions for both, could only proceed at three-fourths of that speed. At what point on the road did they meet?

8. Divide the number 100 into four such parts that the

first may be equal to half the second, but greater than the third by 6, and less than the fourth by 1.

9. Find two numbers such that their sum may be 19, and the difference of their squares 95.

10. What two numbers are those whose sum exceeds their difference by 7, and whose difference exceeds the smaller of the two by two-thirds?

11. Half the difference of two numbers added to half the sum is 93·5, and half the difference taken from half the sum is 77·4. What are the two numbers?

12. Two servants lived together 40 years in the service of the same master, and at the same wages,—viz., £12 per annum. The one spent 10 shillings less every year than the other, and found, at the end of the 40 years, that she had saved twice as much as her fellow-servant, and half-a-year's wages besides. What did they severally save in the year?

13. Two foolish young gentlemen began to play against each other with equal sums of money. After playing 12 hours, and incurring 15 shillings each of expenses, the one remained with twice as much money as the other, having gained from him £15. What had each when they began?

14. On examining a meteorological table for a particular locality, it was found that the whole fall of rain for the year was 22·8 inches,—that the quantity which fell in the first and last quarters together was just equal to that which fell in the second and third,—that the fall in the first quarter was two-thirds of that in the fourth,—and that the third exceeded the second by 2 inches. What were the depths of fall in the respective quarters?

15. A lady, going to purchase a carpet, found, that if she took the best quality in the shop, at 8 shillings the yard, it would take £1 more than the money she brought with her. She therefore contented herself with the second, at 7s. 6d., and found that she had then £2 left. What did the carpet cost?

16. Divide 100 into two such parts, that the excess of the first above 49, shall be to the excess of 49 above the second, as 12 to 11.

17. A boy bought 90 apples and pears for 2s. 4d., having got 3 apples for a penny, and 7 pears for twopence. How many had he of each kind?

18. The difference of two numbers is 40, and the difference of their square roots 2. What are the numbers?

19. Mr Johnson, Mr Thomson, and Mr Wilson, being

one day in company, were requested to contribute to a charitable purpose. Mr Thomson said, that if Mr Johnson subscribed more than £50, he would give, for his subscription, three times as much as all that Mr Johnson's exceeded that sum; while Mr Wilson offered to table four times as much as Mr Thomson's exceeded £50. On hearing this, Mr Johnson, resolving to strain the liberality of his friends, went to the utmost extent of his ability; and Mrs Ellison, who requested their contributions, had the pleasure of departing with £730. What was Mr Johnson's subscription?

20. A *young* sportsman returning unsuccessful from a day's shooting, and meeting his friend with six brace of black game, offered him his gun, shot-belt, and powder-flask for 5 brace; or the gun and belt for 4 brace; or the belt and powder-flask for 3 brace; or the belt, flask, and 6 guineas for the whole. What values did *he* put upon the different articles and the game?

21. Two brothers went from their house to the neighbouring town, taking different roads. The two roads were of equal lengths, but the one was over level ground and the other over a hill,—so that the elder brother, who travelled along the level road, went at the uniform rate of 3 miles an hour; while the younger could advance at only two-thirds of that rate for the first half of the way, thinking to make up for it by doubling his speed the other half. It turned out, however, although he did double his speed, that he was ten minutes longer in reaching his destination. What was the distance of the house from the town?

22. A gentleman bought separately two contiguous fields, containing together 40 acres. Each of these cost as many pounds per acre as there were acres in the field; and the prices of the two fields were in the proportion of 4 to 9. What were the areas of the two fields?

23. A number, consisting of two digits, is reversed, and the reverse subtracted from the original number. It is then found, that the remainder is less than half the given number by 6, but equal to three-fourths of the number reversed. What is the number?

24. On making up the roll of an army after a battle, it was found that the number of effective men was only 714 more than half the number before the battle. Of the number lost, the wounded were twice as many as the slain, and the prisoners equal to one-third of all that remained *for immediate* service, while the number of wounded ex-

ceeded the number of prisoners by 677. What was the original strength of the army?

25. Two railway trains were despatched from the same station, and in the same direction. The second started when the electric telegraph announced that the first was just leaving a station 30 miles in advance, and set out at such a speed that it went 5 miles to the other's 3. After continuing all the 30 miles at that rate, it diminished its speed by 14 miles an hour, making its rate now to that of the train before it as 6 to 5. Suppose both should continue afterwards at the same velocities, and have no stoppages, how long would the second train be, after its first starting, before it overtook the first train?

26. A traveller set out from Edinburgh to London, a distance of 389 miles,—advancing, for the first hundred miles, at the rate of $7\frac{1}{2}$ miles per hour, and for the remainder at 8 miles. At the same time, another traveller set out from London to Edinburgh,—proceeding at the rate of 10 miles per hour for the first hundred miles, and the rest of the way at 9 miles. Find at what distance from London they will meet.

EXERCISES IN CHAPTER XVI, PROBLEM I.

(1.) Given $9x^2 - 126 = 99$.

(2.) ... $\dfrac{25x^2 + 147x}{49} - 10x = 100 - 7x$.

(3.) ... $\dfrac{7x^2}{12} - 7x + 21 = \dfrac{7}{12}$.

(4.) ... $y^2 : 5y - 15 :: 5y - 15 : 16$.

(5.) ... $\dfrac{24x - 49}{5x^2} + \dfrac{46}{5x} = 5$.

EXERCISES IN CHAPTER XVI, PROBLEM II.

(1.) Given $x^2 + 10x = 144$.

(2.) ... $y^2 - 15y + 55 = 1$.

(3.) ... $3x^2 - 18x + 25 = 190$.

(4.) ... $\dfrac{z^2 - 3z + 2}{4} = \dfrac{1}{9}$.

(5.) Given $\frac{x^2}{9} - \frac{4x}{5} = 13.$

(6.) .. $\frac{7x+2}{5} - \frac{2x+4}{3} - \frac{12}{x} + \frac{4x-1}{15} = 0.$

(7.) ... $2y+2 : 4y-10 :: 3y-9 : 3y+6.$

(8.) ... $4 - x - \frac{4x-11}{2x-3} = \frac{x-1}{3}.$

(9.) ... $\frac{6x^2-5x}{2} - \frac{3x^3+5x^2-17x-26}{x-1} = -47.$

(10.) ... $\frac{z-2}{z+2} + \frac{z-4}{z+4} = \frac{119}{78}.$

(11.) ... $\frac{x^2-7x-3}{x-2} - \frac{7x}{15} + \frac{2x-3}{9} - 5 = 0.$

(12.) ... $16x^4 - 12x^2 + 2 = 0.$

(13.) ... $5x^6 - 120x^3 - 405 = 0.$

(14.) ... $x^8 - 10x^4 = 96.$

(15.) ... $x - 12\sqrt{x} - 45 = 0.$

(16.) ... $x - \sqrt{x} = 1.$

Exercises in Chapter XVI, Problem III.

(1.) Given $\left\{\begin{array}{l} 8x^2 - 5y^2 = 20, \\ 6x^2 - 3y^2 = 42. \end{array}\right.$

(2.) ... $\left\{\begin{array}{l} x-y : y :: 1 : 4, \\ xy = 180. \end{array}\right.$

(3.) ... $\left\{\begin{array}{l} x^2 + y^2 = 1 + \frac{1}{144}, \\ xy = \frac{1}{2}. \end{array}\right.$

(4.) ... $\left\{\begin{array}{l} x : x-y :: 22 : 15, \\ x^2 - xy = 36\frac{2}{3}. \end{array}\right.$

(5.) ... $\left\{\begin{array}{l} (y-2z)^2 : (y+2z)^2 :: 1 : 25, \\ yz = 48. \end{array}\right.$

(6.) ... $\left\{\begin{array}{l} 5x^2 - 7y^2 = 17\frac{4}{9}, \\ 4x^2 - 6y^2 = 11\frac{7}{9}. \end{array}\right.$

(7.) ... $\left\{\begin{array}{l} x^2 + 9y^2 - 6xy = 256, \\ x^2 - 9y^2 = 352. \end{array}\right.$

(8.) ... $\left\{\begin{array}{l} x^2 + xy = 3276, \\ xy + y^2 = 5005. \end{array}\right.$

(9.) Given $\begin{cases} x^2 - xy = \frac{9}{16}, \\ xy - y^2 = \frac{9}{25}. \end{cases}$

(10.) ... $\begin{cases} x + y = 22, \\ 5xy - 5(x - y) = 610. \end{cases}$

(11.) ... $\begin{cases} 3x - 4z = 17, \\ x^2 + z^2 = 274. \end{cases}$

(12.) ... $\begin{cases} y - z = 1, \\ y^2z^2 + 16yz = 2436. \end{cases}$

EXERCISES IN CHAPTER XVII.

1. What number is that which is less than one-tenth of its square by $7\frac{1}{2}$?

2. What number is that which exceeds the half of its square by $\frac{3}{8}$?

3. Divide 100 into two such parts, that the product of the whole by one of the parts shall be equal to the square of the other part.

4. A horse-dealer bought a number of young horses at a fair, all at the same price, for the sum of £1350. Of these he kept ten, and sold the remainder for £1306, 10s., making 2 guineas of profit on each of the horses sold. How many horses did he buy?

5. A tradesman computes that, during the time he has been in business, he has made £6300 of clear profit. His neighbour, however, who has been 3 years shorter time in business, has made the same sum, owing to his clearing £27 per annum more. How long is it since the first commenced business?

6. If we add 5 to the smaller of two numbers, and subtract it from the greater, the product of the sum and difference will be 84; and if we add 3 to the smaller, and subtract it from the greater, the product of the sum and difference will be 90. What are the numbers?

7. The express railway-train from *A* to *B*, 350 miles apart, goes 10 miles per hour faster than the slow train, and reaches its destination 4 hours sooner. What are their respective rates?

8. Twenty-eight yards of cloth were purchased in two pieces, which differed both in quantity and in quality, and yet cost equal sums (viz., £9, 15s. a-piece), in consequence of each being accidentally rated at as many shillings per yard as there were yards in the other piece. What were the prices of the two pieces per yard?

9. Divide 39 into three parts, such that the difference of their differences may be 4, and the product of the least and greatest, 225.

10. A wholesale shoemaker received an unexpected order for 990 pairs of shoes, to be finished by the end of the month. He found that, if these were divided equally among his men, each would have allotted to him 12 pairs more than he could get finished by the given time at his ordinary rate of working. He therefore engaged 54 more men, and got the work executed by the time specified. How many men had he at first?

11. A farmer, being asked how many sheep he had, replied, that he had 9 more than an exact number of scores, and that, if the square root of the number of sheep were subtracted from half the number of scores, the remainder would be 3. How many sheep had he?

12. Find three numbers, in the proportion of $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$, such that the sum of the squares of the first and third shall be less than twice the square of the second by 584.

EXERCISES IN CHAPTER XVIII, PROBLEM I.

1. What is the eighth term of the arithmetical progression 17, 32, 47, &c.?

2. What is the eleventh term of the progression $25a$, $20a$, $15a$, &c.?

EXERCISES IN CHAPTER XVIII, PROBLEM II.

1. Sum the series $5a$, $5a + 3b$, $5a + 6b$, &c., to seven terms.

2. How far would a person walk before he could pick up 100 pebbles, placed in a straight line, a yard apart, and bring them one by one to the place where he stood, which was in the same straight line and one yard from the first pebble in the row?

EXERCISES IN CHAPTER XIX, THEOREM.

1. In a geometrical progression the first term is $\frac{a}{b}$, the fifth $\frac{2}{3}$, and the sixth $1\frac{1}{2}$. What is the tenth?

2. The first term of a geometrical progression is $3x$, and the fifth term $\frac{4x^5}{3}$. What is the ninth?

3. What is the amount of £200, for 5 years, at 4 per cent. per annum, compound interest?

EXERCISES IN CHAPTER XIX, PROBLEM I.

1. The first term of a geometrical progression being ·0016, and the common ratio $5a$, what is the tenth term?

2. The first term being $\frac{a^4}{16}$, and the common ratio $\frac{2}{a}$, what is the ninth term?

EXERCISES IN CHAPTER XIX, PROBLEM II.

1. What is the sum of the geometrical progression x, xy, xy^2, ..., xy^6, supposing $x=256$, and $y=1\frac{1}{2}$?

2. A man laid by a farthing for his son when he was a year old, a halfpenny the year after, a penny the third year, and so on,—intending to continue doubling the sum every year till he came of age. If he did so, what would be the accumulated amount which the son would receive at the end of his 21st year?

END OF PART FIRST.

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